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S-Shaped Bifurcation of a Singularly Perturbed Boundary Value Problem

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1. INTRODUCTION AND MAIN RESULT

$$\varepsilon^2 u'' = h(u) - \mu x \tag{1.1}$$

$$u(0) = u'(1) = 0, \tag{1.2}$$

where h is an S-shaped curve as in Fig. 1, $\mu > 0$ and $0 < \varepsilon \ll 1$ are parameters.

The nondecreasing solutions of the problem (1.1), (1.2) represent the u-components of the physically interesting stationary states of the system of equations

$$\begin{gathered}\alpha S_t = S_{xx} + \alpha\varepsilon^2 u_{xx} + \alpha(h(u) - S + \mu x)\\ u_t = \varepsilon^2 u_{xx} + h(u) - S + \mu x \\ u(t,0) = u_x(t,1) = 0, \qquad S(t,0) = S_x(t,1) = 0.\end{gathered} \tag{1.3}$$

This system of equations was studied in [3] as a model of pressure driven flow of a viscoelastic fluid in a capillary. The main purpose of the paper was to modify the model [13] of the experimentally observed phenomenon of spurt [17] by a small diffusion term in the constitutive relation. The relevant questions were the dependence of the non-negative solutions of (1.1), (1.2) on the parameter μ (representing pressure) for ε small as well as the stability of the corresponding steady states of the problem (1.3).

In [3], we have shown that for $\varepsilon > 0$ sufficiently small and

- $\mu < \gamma_0$ there is a unique solution of (1.1), (1.2); this solution has neither a boundary nor an internal layer,

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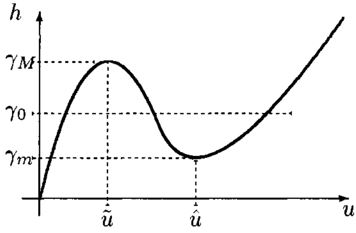

Fig. 1. The function h.

- $\gamma_0 < \mu < \gamma_M$ there are at least three solutions: one without layers, one with an internal layer and one with a boundary layer, and
- $\mu > \gamma_M$ there is a unique solution which has an internal layer

The meaning of γ_m, γ_M is clear from Fig. 1; γ_0 is the "Maxwell" value of the parameter μ satisfying the "equal area rule"

$$\int_{r_1}^{r_2} [h(u) - \mu]\, du \left(= \int_{r_1}^{r_2} h(u)\, du - \mu(r_2 - r_1) \right) = 0,$$

r_1, r_2 being the outer zeros of $u \mapsto h(u) - \gamma_0$. Furthermore, it was shown that local stability of a solution of (1.1), (1.2) with respect to (1.3) is equivalent to the local stability of the reduced problem

$$\begin{aligned} u_t &= \varepsilon^2 u_{xx} + h(x) - \mu x \\ u(t, 0) &= u'(t, 1) = 0 \end{aligned} \tag{1.4}$$

and that the unique solution of (1.1), (1.2) for $\mu < \gamma_0$ and $\mu > \gamma_M$ is globally asymptotically stable.

While these results have been reasonably satisfactory from the physical point of view some challenging mathematical problems have remained open. In this paper, we address one of them—the bifurcation of the solutions of (1.1), (1.2) for a fixed $\varepsilon > 0$.

The results of [3] indicate that bifurcation of the nondecreasing solutions of (1.1), (1.2) on μ for $0 < \varepsilon \ll 1$ fixed can be depicted by a common S-shaped diagram with two folds (Fig. 2).

The goal of this paper is to justify the diagram of Fig. 2 as a one representing all nondecreasing solutions of (1.1), (1.2) for a fixed small $\varepsilon > 0$.

To achieve this goal one has to

- show that the solutions established in [3] are globally the only ones,
- analyze the local bifurcations at $\mu = \gamma_0, \gamma_M$.

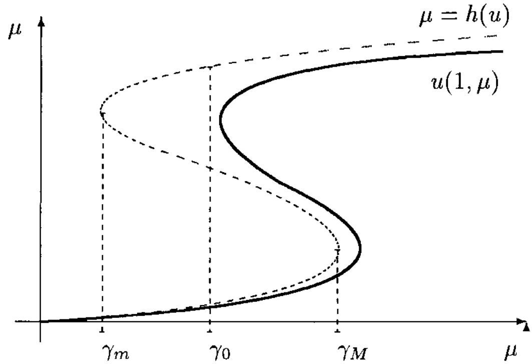

Fig. 2. The bifurcation diagram.

The second problem is more interesting. Its essence lies in the fact that the interval of ε's for which the conclusions of [3] have been derived has been dependent on μ and has degenerated near γ_0 and γ_M (Fig. 3).

For the analysis of the behavior of the solutions of (1.1), (1.2) near the bifurcation points γ_0 and γ_M new techniques had to be employed. For the bifurcation at γ_0 this was in the first place a method going back to [12] which was further developed in [14] and adjusted to singular perturbation problems in [9]. Secondly, a C^2-extension of the Exchange Lemma of [7, 8] was needed. Both these techniques may be of a more general interest for geometric singular perturbation theory in general. In addition to the C^2 Exchange Lemma, for the bifurcation at γ_M a combination of ad hoc methods had to employed. The future will show whether some of them find wider use.

The main result of this section is

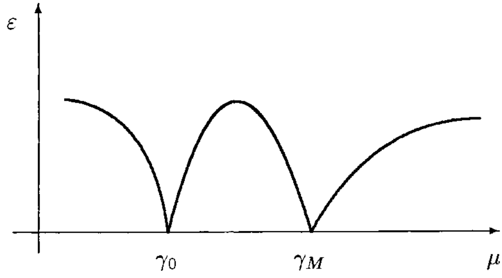

Fig. 3. Estimates on ε.

1.1. THEOREM. *Let* $0 < \tilde{u} < \hat{u}$ *and let* $h: [0, \infty) \to [0, \infty)$ *be a* C^4 *function such that* $h(0) = 0$, $h''(0) < 0$, $h''(\infty) > 0$, $h(u) > 0$ *for* $u > 0$, $h'(u) > 0$ *for* $0 < u < \tilde{u}$, *and* $u > \hat{u}$, $h'(u) < 0$ *for* $\tilde{u} < u < \hat{u}$, $h''(u)$ *having a unique zero which is located between* $\tilde{u}$ *and* $\hat{u}$. *Denote* $\gamma_m = h(\hat{u})$, $\gamma_M = h(\tilde{u})$. *Then, given* $\bar{\mu} > \gamma_M$ *there is an* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *there exist* $\mu_0(\varepsilon)$, $\mu_M(\varepsilon)$ *satisfying* $\mu_0(\varepsilon) \to \gamma_0$, $\mu_M(\varepsilon) \to \gamma_M$ *for* $\varepsilon \to 0$ *such that*

(i) *For* $0 < \mu < \mu_0(\varepsilon)$ *there is a unique nondecreasing solution* u_1 *of* (1.1), (1.2). *This solution has no layers.*

(ii) *For* $\mu_0(\varepsilon) < \mu < \mu_M(\varepsilon)$ *there are three nondecreasing solutions* u_1, u_2, u_3 *of* (1.1), (1.2). *The solution* u_1 *has no layers,* u_2 *has an internal and* u_3 *a boundary layer.*

(iii) *For* $\mu > \mu_M(\varepsilon)$ *there is a unique nondecreasing solution* u_2 *of* (1.1), (1.2). *This solution has an internal layer.*

(iv) u_1 *merges with* u_3 *at* $\mu_M(\varepsilon)$ *and* u_2 *merges with* u_3 *at* $\mu_0(\varepsilon)$ *in a generic fold bifurcation*

By the conclusion (iv) concerning μ_0 we understand that the BVP (1.1), (1.2) can be reduced to a scalar equation for μ and another variable ζ the set of solutions of which is locally a C^2 curve $\mu = \chi(\zeta)$ defined in a neighborhood of ζ_0 such that $\mu(\zeta_0) = \mu_0(\varepsilon)$, $\mu'(\zeta_0) = 0$, $\mu''(\zeta_0) \neq 0$ and $u_1 = \chi(\zeta)$ for $\zeta > \zeta_0$, $u_2 = \chi(\zeta)$ for $\zeta < \zeta_0$; similarly we understand the conclusion concerning μ_M.

The proof, in fact, yields slightly more: The solutions u_1, u_2, u_3 exist and are C^2 functions of μ on $[0, \mu_M)$, $(\mu_0, \bar{\mu}]$, (μ_0, μ_M), respectively. Let us note that u_1 and u_2 represent the u-components of asymptotically stable equilibria of (1.3) while u_3 represents the u-component of an unstable equilibrium. We will not discuss the problems of dynamics of (1.3) further here. Due to the equivalence of the stability properties of (1.3) and (1.4) they can be resolved by rather standard methods.

The rest of this article is organized as follows. Section 2 summarizes the contents of [3] to an extent necessary for the understanding of the article. In Section 3 the C^2-extension of the Exchange Lemma for our particular case is established. This extension of the lemma together with the technique of [9] is used in Section 4 to analyze local bifurcation of the solutions of (1.1), (1.2) at γ_0. Section 5 is devoted to the analysis of the bifurcation at γ_M. By synthesizing the results of Sections 2–5, Theorem 1.1 is proved in Section 6. In Appendix 1 results of geometric singular perturbation theory relevant to this article are summarized. Appendix 2 refers to Section 4 and contains the technically complicated background of the bifurcation analysis of Section 4 in the spirit of [9].

The assumptions posed on h in the theorem as well as the definitions of $\bar{\mu}$, γ_m, γ_0, γ_M will be used throughout the article without further notice.

2. PRELIMINARIES

The essence of singular perturbations techniques of solving the problem (1.1), (1.2) lies in attempts to perturb "singular" solutions. Those are concatenations of arcs satisfying the "fast time" and "slow time" $\varepsilon \to 0$ limits of the Eq. (1.1).

As the "fast time" system associated with (1.1) we consider the equations

$$\begin{aligned} \varepsilon u' &= v \\ \varepsilon v' &= h(u) - \mu x \end{aligned} \tag{2.1}$$

the $\varepsilon \to 0$ limit of which is

$$v = 0, \qquad h(u) - \mu x = 0. \tag{2.2}$$

The "slow time" system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= h(u) - \mu x \\ \dot{x} &= \varepsilon \end{aligned} \tag{2.3}$$

is obtained from (2.1) by the time scale change $dx = \varepsilon dt$. Its $\varepsilon \to 0$ limit is a family of planar systems

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= h(u) - \xi \end{aligned} \tag{2.4_ξ}$$

in which $\xi = x/\mu$ appears as a parameter.

Thus, by a singular solution curve we understand a finite continuous concatenation of continuous arcs $\Sigma_1, ..., \Sigma_n$ in the (x, u, v)-space such that:

(Σ1) Σ_i is either an arc of points solving (2.2) parametrized by increasing $x \in [0, 1]$ or an arc with $x \in [0, 1]$ fixed. In the latter case (u, v) is a part of a trajectory of ($2.4_{\mu x}$) parametrized by increasing t.

(Σ2) If Σ_i is an arc solving (2.2) parametrized by $x \in [x_1, x_2]$ then its neighbors are arcs with $x = x_1$, $x = x_2$ and u, v solving ($2.4_{\mu x_1}$), ($2.4_{\mu x_2}$), respectively.

(Σ3) The u, x components of the initial point of Σ_1 and the v-component of the terminal point of Σ_n vanish. The x-component of the terminal point of Σ_n is 1.

It is instructive for the future to keep in mind the phase portraits of the system (2.4_ξ) for various values of ξ (Fig. 4).

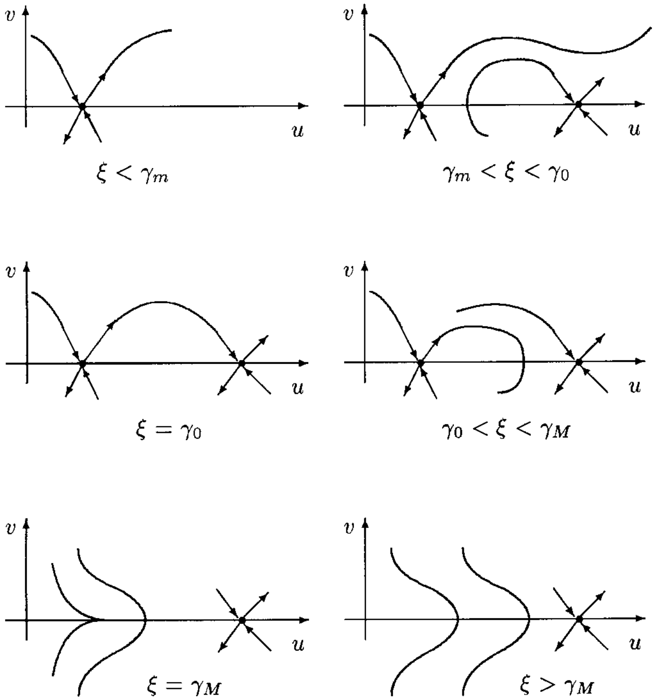

Fig. 4. Phase portraits of (2.4_ξ).

We summarize the important properties of the Eq. (2.4_ξ) in its dependence on ξ:

(P1) For $\xi < \gamma_m$ there is a unique equilibrium of (2.4_ξ) to be denoted by $(r_1(\xi), 0)$. This equilibrium is a hyperbolic saddle and a C^4-function of $\xi \in (0, \gamma_M)$.

(P2) At $\xi = \gamma_m$ two additional equilibria $(r_3(\xi), 0)$ and $(r_2(\xi), 0)$ with $r_3(\xi) < r_2(\xi)$ are created by a generic saddle-node bifurcation. The equilibrium $r_3(x)$ is a center, $r_2(\xi)$ is a hyperbolic saddle for $\gamma_m < \xi < \gamma_M$. While $r_2(\xi)$ can be continued as a C^4 function beyond γ_M, $r_3(x)$ merges with $r_1(\xi)$ at $\xi = \gamma_M$ in another generic saddle node bifurcation.

(P3) At $\xi = \gamma_0$ there is a heteroclinic trajectory $K := \{(u_0(t), v_0(t)) : t \in \mathbb{R}, (u_0(t), v_0(t))$ a solution of $(2.4_{\gamma_0})\}$.

For $\eta \geqslant 0$ we denote

$$M_1(\eta) = \{(\xi, r_1(\xi), 0) \colon \xi < \gamma_M - \eta\}$$

$$M_2(\eta) = \{(\xi, r_2(\xi), 0) \colon \xi > \gamma_m + \eta\}$$

as well as

$$M_i = M_i(0), \qquad i = 1, 2.$$

By Propositions A1.1–A1.3, for each $\eta > 0$ there exist C^4 submanifolds $\mathbf{M}_i(\eta) = 1, 2$ of $\mathbb{R}^4$ locally invariant with respect to (2.3) augmented by $\dot{\varepsilon} = 0$ such that:

(M1) $\mathbf{M}_i(\eta) \cap \{\varepsilon = 0\} = M_i(\eta)$.

(M2) The natural x-projections of the manifolds

$$M_i^\varepsilon(\eta) = \mathbf{M}_i(\eta) \cap \{\varepsilon \text{ fixed}\}$$

contain the intervals $[0, \gamma_M - \eta]$ for $i = 1$ and $[\gamma_m + \eta, \mu]$ for $i = 2$,

(M3) For $i = 1, 2$, $\mathbf{M}_i(\eta)$ admits a stable manifold $W_i^s(\mathbf{M}_i(\eta))$ and an unstable manifold $W_i^u(\mathbf{M}_i(\eta))$, both locally invariant, such that

$$W_1^s(\mathbf{M}_i(\eta)) \cap \{\varepsilon = 0\} = W^s(M_i(\eta))$$

$$W_i^u(\mathbf{M}_i(\eta)) \cap \{\varepsilon = 0\} = W^u(M_i(\eta)),$$

the solutions on which are uniformly exponentially attracted resp. repelled by $\mathbf{M}_i(\eta)$. As a consequence of (i) and (iii) we obtain that:

(M4) $M_i^\varepsilon(\eta)$ and compact parts of $W^j(M_i^\varepsilon(\eta))$, $i = 1, 2$, $j = u, s$ are uniformly $O(\varepsilon)$ C^2-close to $M_i(\eta)$, $W^j(M_i(\eta))$, respectively.

(M5) Moreover, for $i = 1$ or 2, by a C^3 change of coordinates, the system (2.3) can be in the neighbourhood of $\mathbf{M}_i(\eta)$ brought to the form

$$\begin{aligned} \dot{a} &= A(a, b, x, \varepsilon)\, a \\ \dot{b} &= B(a, b, x, \varepsilon)\, b \\ \dot{x} &= \varepsilon, \end{aligned} \tag{2.5}$$

where A, B are C^2 satisfying

$$A(a, b, x, \varepsilon) > \alpha > 0; \tag{2.6}$$

$$B(a, b, x, \varepsilon) < -\beta < 0. \tag{2.7}$$

In (2.5) one has

$$\mathbf{M}_i(\eta) = \{a=0, b=0\},$$
$$W^u(\mathbf{M}_i(\eta)) = \{b=0\},$$
$$W^s(\mathbf{M}_i(\eta) = \{a=0\},$$

locally in a, b near $a=b=0$. In case of no need to specify η we will drop it as argument

From Fig. 4, or, (Σ1)–(Σ3) and (P1)–(P3) it follows that an arc Σ of a trajectory of (2.4_ξ) can be a part of a singular solution curve in the following cases only:

(i) $\mu \geqslant \gamma_0$, $\xi = \mu x = \gamma_0$ and Σ is the heteroclinic trajectory K connecting $r_1(\gamma_0, 0)$ to $r_2(\gamma_0, 0)$,

(ii) $\gamma_0 < \mu < \gamma_M$, $x = \xi/\mu = 1$ and $\Sigma = \Sigma_n$ is the $v \geqslant 0$ part of the unstable separatrix of $(r_1(\xi), 0)$ to be denoted by $H(\xi)$.

Consequently, depending on the parameter μ, we have the following non-negative singular solution curves:

(S1) $u = r_1(\mu x)$, $v=0$ with no layers for $\mu < \gamma_M$,

(S2) the concatenation of the arc $u = r_1(\mu x)$, $v=0$, $0 \leqslant x \leqslant \gamma_0/\mu$, the heteroclinic trajectory K (internal layer) and the arc $u = r_2(\mu x)$, $v=0$, $\gamma_0/\mu \leqslant x \leqslant 1$ for $\mu > \gamma_0$,

(S3) the concatenation of the arc $u = r_1(\mu x)$, $v=0$, $0 \leqslant x \leqslant 1$ and the arc $H(\mu)$ of the unstable separatrix of $(r_1(\mu), 0)$ (boundary layer) for $\gamma_0 < \mu < \gamma_M$.

The following proposition [3] holds true.

2.1. Proposition. *Given $\delta > 0$ and $d > 0$ sufficiently small there exists an $\varepsilon > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$ and $\mu \in [0, \mu_M - \delta]$, $\mu \in [\gamma_0 + \delta, \bar{\mu}]$, $\mu \in [\gamma_0 + \delta, \gamma_M - \delta]$ there is a solution u of the BVP* (1.1), (1.2) *whose trajectory* $(u(x), v(x))$ *of* (2.1) *is in the d-neighborhood of the singular solution curve* (S1),(S2),(S3), *respectively.*

We will prove Proposition 2.1 as a part of Theorem 1.1 in Section 6. The following facts will be essential for the proof:

(B1) The left hand boundary condition manifold $u=0$ intersects the stable separatrix of the equilibrium $(r_1(0), 0)$ of (2.4_0) transversally.

(B2) The right hand boundary condition manifold $v=0$ intersects the unstable separatrices of the points $(r_1(\mu), 0)$ resp. $(r_2(\mu), 0)$ transversally provided $\mu < \gamma_M$ resp. $\mu > \gamma_m$.

(B3) The manifold $W^u(M_1)$ intersects $W^s(M_2)$ transversally along the heteroclinic K: $u=u_0(t)$, $v=v_0(t)$, upside down with respect to v for ξ increasing: If Π is any hyperplane $u=\text{const.}$ intersecting K at a point (u_0, v_0) and, if $(1, y)$ and $(1, z)$ are the vectors generating $TW^u(M_1)\cap\Pi$ and $TW^s(M_2)\cap\Pi$ respectively, then $y<z$.

The transversality condition (B3) is satisfied since the Melnikov integral

$$\int_{-\infty}^{\infty}\left\langle \psi_2(t), \frac{\partial}{\partial\xi}(h(\hat{u}(t))-\xi)\right\rangle\bigg|_{\xi=\gamma_0} dt=\int_{-\infty}^{\infty}\psi_2(t)\,dt<0, \tag{2.8}$$

where $\psi(t)$ is the unique (up to the positive multiplicative constant) bounded solution of the adjoint equation

$$\dot{\psi}_1=-h'(u_0(t))\,\psi_2, \qquad \dot{\psi}_2=-\psi_1$$

of the linearization of (2.3_{γ_0}) along $(u_0(t), v_0(t))$ such that $\psi_2(0)>0$. To see why the inequality (2.8) holds note that $\psi_2(t)$ is a positive multiple of $\dot{v}_0(t)$, hence $\psi_2(t)>0$ for all t [15].

We denote by φ_t the flow of (2.3), $P_u, P_v, P_{uv}, \ldots$ the natural projections of the (u, v, x) space into its $u-$, $v-$, $(u, v)-$, ... subspaces, respectively. For a subset $Q\subset\mathbb{R}^2$ we denote

$$\Phi_{x,\xi}(Q)=\{P_{uv}\varphi_{(x-\xi)/\varepsilon}(u, v, \xi): (u, v)\in Q\}.$$

In terms of system (2.1), $\Phi_{x,\xi}(Q)$ consists of the values $u(x)$, $v(x)$ of its solutions satisfying $(u(\xi), v(\xi))\in Q$. In particular, we denote $\Gamma=\{u=0\}$, $\Lambda=\{v=0\}$ and

$$\begin{aligned} \Gamma(x)&=\Phi_{x,0}(\Gamma), & \Lambda(x)&=\Phi_{x,1}(\Lambda),\\ \mathbf{\Gamma}&=\bigcup_x \Gamma(x), & \mathbf{\Lambda}&=\bigcup_x \Lambda(x). \end{aligned} \tag{2.9}$$

Obviously, $u(x)$ is a solution of (1.1), (1.2) if and only if $(u(x), u'(x))\in \Gamma(x)\cap\Lambda(x)$ for all $x\in[0, 1]$.

3. A C^2 INCLINATION LEMMA

In this section we consider the system

$$\begin{aligned} \dot{x}&=\varepsilon\\ \dot{y}&=f(x, y, \varepsilon), \end{aligned} \tag{3.1}$$

with $x \in \mathbb{R}$, $y \in \mathbb{R}^2$, f being C^4. We assume that for $0 \leqslant x \leqslant X$ the equation

$$f(x, y, 0) = 0$$

admits a continuous curve $M: y = r(x)$ of solutions such that $D_y f(x, r(x), 0)$ has a pair of real nonzero eigenvalues with different signs. As we know from Proposition A1.1, in such a case there exists a two-dimensional invariant manifold $\mathbf{M} \subset \mathbb{R}^4$ of (3.1) augmented by $\dot{\varepsilon} = 0$ the $\varepsilon -$section M^ε of which is M for $\varepsilon = 0$. Moreover, for $\varepsilon \geqslant 0$ sufficiently small each invariant manifold M^ε admits an unstable manifold $W^u(M^\varepsilon)$ and a stable manifold $W^s(M^\varepsilon)$. We prove

3.1. Proposition. *Let* $0 \leqslant \underline{x} < x^* < X$, $K = [\underline{x}, X] \times K_1$, K_1 *a compact interval of* $\mathbb{R}$ *and let* Ψ^ε *be the diffeomorphism* $K \times I \to W^\varepsilon$ *defined in Corollary* A1.4 *with* $M^\varepsilon \cap \{\underline{x} \leqslant x \leqslant X\} \subset \Psi^\varepsilon(K \times \{0\})$. *Let* Σ^ε, $0 \leqslant \varepsilon \leqslant \varepsilon_0$ *be a family of* C^2*-curves intersecting* $W^s(M^\varepsilon)$ *uniformly transversally at a point of* $W^\varepsilon \cap \{x = \underline{x} + O(\varepsilon)\}$. *Then, the manifold* N^ε *formed by the* $x^* \leqslant x \leqslant X$ *parts of the trajectories of* (3.1) *in* W^ε *through the points of* Σ^ε C^2 $O(e^{-\rho/\varepsilon})$ *approximates* $\Psi^\varepsilon(K) \cap \{x \geqslant x^*\}$ *for some* $\rho > 0$, *i.e., there is a function* $s^\varepsilon: K \to I$ *such that*

$$N^\varepsilon = \Psi^\varepsilon(\text{graph } s^\varepsilon)$$

and

$$|D^j s^\varepsilon| \leqslant O(e^{-\rho/\varepsilon})$$

for $j = 0, 1, 2$, *uniformly over* K.

The meaning of the assumption of uniform transversality will be made clear in Lemma 3.4 below.

3.2. Corollary. *Because of Proposition* A1.3, *Proposition* 3.1 *holds true with* M^ε *replaced by* M *and* $O(e^{-\rho/\varepsilon})$ *replaced by* $O(\varepsilon)$.

3.3 *Remark.* Note that in Proposition 3.1 and Corollary 3.2 trajectories of points of Σ^ε can leave W^ε through the lateral boundary $\Psi(\partial K \cap I)$ only.

Proposition 3.1 follows from the lemma below in the same way as Proposition A1.3 does from Proposition A1.2. The lemma concerns the local behavior of N^ε near M^ε. In it, we consider a system of equations

$$\begin{aligned} \dot{a} &= A(a, b, x, \varepsilon)\, a, \\ \dot{b} &= B(a, b, x, \varepsilon)\, b, \\ \dot{x} &= \varepsilon, \end{aligned} \tag{3.2}$$

where $a, b, x \in \mathbb{R}$, A, B are C^2 satisfying (2.6), (2.7) for $(a, b, x, \varepsilon) \in \Omega = \{|a| \leqslant 3\Delta,\ |b| \leqslant 3\Delta,\ \underline{x} \leqslant x \leqslant X,\ 0 \leqslant \varepsilon \leqslant \varepsilon_0\}$, $\Delta, \varepsilon_0 > 0$. By Proposition A1.3, the system (3.1) can be locally at M^ε written in the form (3.2) with $W^u_{loc}(M^\varepsilon) = \{b = 0\}$ and $W^s_{loc}(M^\varepsilon) = \{a = 0\}$

We denote by ϕ_t the (local) flow of (3.2) and by $P_a, P_b, ...; P_{ax}, ...$ the natural projections of $\mathbb{R}^3$ to the planes $b = x = 0, a = x = 0, ..., b = 0, ...,$ respectively.

3.4. Lemma. *Let* $x^* \in (\underline{x}, X]$, *let* $0 \in I \subset \mathbb{R}$ *be open and let* $\sigma^\varepsilon \in C^2([-\Delta, \Delta], [-\Delta, \Delta] \times \mathbb{R})$ *be a family of functions satisfying*

$$P_x \sigma^\varepsilon(0) = \underline{x} + O(\varepsilon) \tag{3.3}$$

and the (*uniform transversality*) *condition*

$$|D^j \sigma^\varepsilon(a)| \leqslant \Lambda \tag{3.4}$$

for some $\Lambda > 0$, $j = 1, 2$ *and all* $a \in [-\Delta, \Delta]$, $0 \leqslant \varepsilon \leqslant \varepsilon_0$. *Let* N^ε *be the locally invariant manifold of* (3.2) *defined by*

$$\begin{aligned} N^\varepsilon = \{\phi_t(a, b, x) \colon (a, b, x) \in \Sigma^\varepsilon,\ x^* \leqslant \varepsilon t \leqslant X, \\ |P_a \phi_\tau(a, b, x)| \leqslant 3\Delta \text{ for } 0 \leqslant \tau \leqslant t\}, \end{aligned} \tag{3.5}$$

where $\Sigma^\varepsilon = \text{graph}\, \sigma^\varepsilon$. *Then, for* $0 < \varepsilon \leqslant \varepsilon_0$ *sufficiently small there exists a function* $s^\varepsilon \in C^2([-\Delta, \Delta] \times [x^*, X], \mathbb{R})$ *such that*

$$N^\varepsilon = \text{graph}\ s^\varepsilon.$$

One has

$$|D^j s^\varepsilon(a, x)| = O(e^{-\varrho/\varepsilon}), \qquad j = 0, 1, 2 \tag{3.6}$$

for some $\rho > 0$, *uniformly in* a, x.

Note that in the definition (3.5) of N^ε we consider the local flow ϕ to be strictly confined to Ω. That is, points on trajectories through graph σ^ε which have left Ω between 0 and t are not included into N^ε even if ϕ could be extended beyond the limits of Ω.

Lemma 3.4 is a version of the "Exchange Lemma" of [8] in our simple special situation. The information about N^ε given by it is more precise compared to the one obtained by a direct application of the result of [8] in two ways: It asserts that $P_a \colon N_\varepsilon \to P_a N_\varepsilon$ is a global isomorphism and, in addition, that N^ε converges to the plane $b = 0$ in a C^2 way.

The general version of Lemma 3.4 will be the subject of a forthcoming paper.

Proof. To simplify notation rescale x in such a way that $\underline{x} = 0$, $x^* = 1$ and drop the superscript ε at N, σ, s.

A point (a, b, x) belongs to N if and only if $|a| \leqslant \Delta$, $|b| \leqslant \Delta$, $1 \leqslant x \leqslant X$ and

$$a = P_a \phi_T(v, \sigma(v)), \tag{3.7}$$

$$b = P_b \phi_T(v, \sigma(v)), \tag{3.8}$$

$$x = P_x \sigma(v) + \varepsilon T, \tag{3.9}$$

for some $|v| \leqslant \Delta$. To prove that s is well-defined and satisfies (3.6) we first show that (3.7), (3.9) have a unique solution in v, T for $|a| \leqslant \Delta$, $1 \leqslant x \leqslant X$ and estimate its derivatives.

Denote $\mathscr{C} = \{u = (a, b) \in C([0, T], \mathbb{R}^2) : |a(t)|, |b(t)| \leqslant 3\Delta \text{ for } 0 \leqslant t \leqslant T\}$. A solution $u(t) = (a(t), b(t), x(t))$ of (3.2) satisfies $b(0) = w$, $x(0) = \xi$, $a(T) = a$ if and only if it is the fixed point of the nonlinear integral operator $\mathscr{F} : [-\Delta, \Delta] \times [-\Delta, \Delta] \times [0, 1] \times \mathscr{C} \to \mathscr{C}$ in the third argument, defined by

$$\mathscr{F}(a, w, \xi, u)(t) = e^{\mathscr{A}(t, T, \xi)} a + \int_T^t e^{\mathscr{A}(t, s, \xi)} f(u(s), \xi + \varepsilon s)\, ds$$
$$+ e^{\mathscr{B}(t, 0, \xi)} w + \int_0^t e^{\mathscr{B}(t, s, \xi)} g(u(s), \xi + \varepsilon s)\, ds,$$

where

$$\mathscr{A}(t, \tau, \xi) = \int_\tau^t A(0, 0, \xi + \varepsilon s)\, ds, \qquad \mathscr{B}(t, \tau, \xi) = \int_\tau^t B(0, 0, \xi + \varepsilon s)\, ds,$$

$$f(a, b, x) = [A(a, b, x) - A(0, 0, x)]\, a,$$

$$g(a, b, x) = [B(a, b, x) - B(0, 0, x)]\, b.$$

Note that $f(0, 0, x) = 0$ and that $L = \sup_\Omega \{|Df|, |Dg|\}$ can be made arbitrary small by the choice of a sufficiently small Δ.

By $\|\cdot\|$ we denote the norm in $C([0, T], \mathbb{R}^2)$. We have

$$\|\mathscr{F}(a, w, \xi, u)\| \leqslant |a| + |w| + L(1/\alpha + 1/\beta)\, \|u\|.$$

Hence, $\mathscr{F}(a, w, \xi, \cdot)$ maps $\mathscr{C}$ into itself provided

$$L(1/\alpha + 1/\beta) \leqslant 1/3 \tag{3.10}$$

Moreover, if (3.10) is satisfied, $\mathscr{F}(a, w, \xi, \cdot)$ is a contraction with contraction constant 1/3. Since f, g are C^2, by the uniform contraction principle [4] $\mathscr{F}$ has a unique fixed point $u = \Psi(a, w, \xi, T)$ which is a C^2 function of

a, w, ξ with bounded derivatives. The fixed point is a C^2 function of T as well. This does not follow immediately from the principle since variation of T changes the space $\mathscr{C}$. However, it can be easily reduced to the case of $\mathscr{C}$ fixed by rescaling time as in [9], Proposition 4. Since $u(t) = \Psi(a, w, \xi, T)(t)$ solves (3.2), $P_a u(t)$ is increasing and $P_b u(t)$ is decreasing. Hence,

$$\begin{aligned} \|P_a \Psi(a, w, \xi, T)\| &\leqslant |P_a \Psi(a, w, \xi, T)(T)| = |a| \leqslant \Delta, \\ \|P_b \Psi(a, w, \xi, T)\| &\leqslant \Delta. \end{aligned} \tag{3.11}$$

Now, (v, T) solves (3.7), (3.9) if and only v is a fixed point of the operator

$$G(a, x, v) = P_a \Psi(a, \sigma(v), \varepsilon^{-1}(x - P_x \sigma(v)))(0)$$

in v and $P_x \sigma(v) + \varepsilon T = x$. By (3.3), we have

$$|G(a, x, v)| \leqslant e^{-\alpha/(2\varepsilon)} \Delta + L \int_T^0 e^{\mathscr{A}(0, s)}\, ds\, \Delta = O(e^{-1/(2\varepsilon)} + L/\alpha)\, \Delta \leqslant \Delta,$$

and, due to (3.4),

$$|D_v G(a, x, v)| \leqslant O\left(\frac{1}{\varepsilon} e^{-\alpha/(2\varepsilon)} (1 + L)\, \Lambda + \Lambda L/\alpha\right) < 1/2,$$

provided Δ and ε_0 have been chosen sufficiently small.

Thus, for Δ and $0 < \varepsilon \leqslant \varepsilon_0$ small enough, G maps $\{|v| \leqslant \Delta\}$ uniformly contractively into itself. Therefore, it has a unique fixed point $v = h(a, x)$ in $\{|v| \leqslant \Delta\}$ which is a C^2 function of a, x. The uniform contraction principle also provides formulas for the derivatives of h and Ψ. One has

$$D_k h = (I - D_v G)^{-1} D_k G, \tag{3.12}$$

$$D_{kl}^2 h = (I - D_v G)^{-1} [D_{vv} G D_k h D_l h + D_{vk} G D_l h + D_{vl} G D_k h + D_{kl} G] \tag{3.13}$$

for $k, l = a, x$; similar formulas hold for the derivatives of Ψ. Thus, for $x \geqslant 1$ we have

$$\begin{aligned} |D_a h(a, x)| &\leqslant 2\, |D_a P_a \Psi(a, \sigma(v), \varepsilon^{-1}(x - P_x \sigma(v)))(0)|_{v = h(a, x)} \\ &\leqslant 2 e^{\mathscr{A}(0, T)} = O(e^{-\alpha(x - 1/2)/\varepsilon}) = O(e^{-\alpha/(2\varepsilon)}) \end{aligned}$$

provided ε_0 and Δ are so small that for $O(\varepsilon)$ from (3.3) one has $O(\varepsilon) + \Lambda \Delta < 1/2$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Similarly, we obtain

$$|D_x h(a, x)| = O(\varepsilon^{-1} A(0, 0, x)\, e^{-\alpha/(2\varepsilon)}).$$

It should now be obvious that each term of (3.13) involves at least one factor of order $e^{-\alpha x/(2\varepsilon)}$, the growths of the others being at most ε^{-2}. Thus,

$$|D_j h| = O(e^{-\beta/\varepsilon}) \tag{3.14}$$

for $x \geqslant 1$ and $j = 1, 2$ with some $0 < \beta < \alpha/2$.

Now, we define

$$s(a, x) = P_b \Psi(a, \sigma(h(a, x)), \varepsilon^{-1}(x - P_x \sigma(h(a, x))))(T).$$

Obviously, (3.7)–(3.9) are satisfied if and only if $v = h(a, x)$ and $b = s(a, x)$. The estimates (3.6) follow from (3.4) and (3.14) by a straightforward application of the chain rule formula. ∎

4. THE BIFURCATION AT $\mu = \gamma_0$

To study the bifurcation we employ a method going back to [12] which has been refined in [14] and adjusted to singular perturbation problems in [9].

To this end, by the transformation

$$\mu x = \xi, \qquad \varepsilon\mu = \nu \tag{4.1}$$

we transform the BVP (1.1), (1.2) to the form

$$\nu^2 u'' = h(u) - \xi \tag{4.2}$$

$$u(0) = u'(\mu) = 0, \tag{4.3}$$

the parameter now being the length of the interval. We fix $\eta > 0$ in such a way that $r_1(\gamma_0) \in M_1(\eta)$ and $r_2(\gamma_0) \in M_2(\eta)$ and drop η until the end of this section. Referring to (2.3) with $\mu = 1$ and x, ε replaced by ξ, ν, respectively, we define M_i, Γ, etc. as in Section 2.

Following (M5) of Section 2 we introduce Fenichel's coordinates at the manifolds M_2^ν perturbing M_2. After changing the time scale by $d\xi = \nu dt$, in those coordinates (4.2) locally near M_2^ν transcribes to the system

$$\dot{a} = A(a, b, \xi, \nu) a \tag{4.4}$$

$$\dot{b} = B(a, b, \xi, \nu) b \tag{4.5}$$

$$\dot{\xi} = \nu. \tag{4.6}$$

Recall that $M_2^\nu = \{a = 0, b = 0\}$, $W^u_{loc}(M_2^\nu) = \{b = 0\}$, $W^s_{loc}(M_2^\nu) = \{a = 0\}$.

Let $\Delta > 0$, $v_0 > 0$ be such that (4.4)–(4.6) represents (4.2) and let

$$\alpha > A > \alpha > 0, \quad \bar{\beta} > -B > \beta > 0$$

for $|a| \leqslant \Delta$, $|b| \leqslant \Delta$, $v \leqslant v_0$, $|\xi - \gamma_0| < 2\kappa$.

For $v = 0$ (i.e., $\varepsilon = 0$), the surfaces $a = 0, b = 0$ are foliated by the stable resp. unstable separatrices of the saddle points $(r_2(\mu), 0)$ constituting M_2. We orient a, b in such a way that the $v < 0$ part of the unstable separatrix of $(r_2(\mu), 0)$ points to the $a > 0$ halfspace while the $v > 0$ part of the stable separatrix points to the $b > 0$ halfspace. With this orientation, the right hand boundary condition line $v(\mu) = 0$ (to be denoted by $L_0(\mu)$) passes through the points $(r_2(\mu), 0)$ bisecting the angle of the separatrices of the latter for $v = 0$. In particular, we have

$$L_0(\mu) = \{a = n_0(\mu)\, b + \Omega_0(b, \mu)\}$$

where $n_0(\mu) > 0$ for μ near γ_0, $\Omega_0(0, \mu) = D_b \Omega_0(0, \mu) = 0$.

It follows from Proposition A1.1 that for $v \geqslant 0$ the boundary condition line $v(\mu) = 0$, to be denoted by $L_v(\mu)$, is given by

$$L_v(\mu) = \{a = m(\mu, v, b)\, v + n(\mu, v)\, b + \Omega_v(b, \mu)\}, \tag{4.7}$$

where

$$\Omega_v(0, \mu) = D_b \Omega_v(0, \mu) = 0,$$

hence

$$D_b^j \Omega_v(b, \mu) = O(b^{2-j}), \qquad D_\mu^j \Omega_v(b, \mu) = O(b^2) \tag{4.8}$$

for $j = 0, 1, 2$.

4.1. Lemma. *We have*

$$m(\mu, 0, 0) > 0 \tag{4.9}$$

for μ near γ_0.

Proof. Since $n_0(\mu) > 0$, geometrically (4.9) means that the boundary line $L_v(\mu)$ intersects the $b = 0$, $a = 0$ axes at points with $a > 0$, $b < 0$, respectively. This, in turn, is equivalent to

$$v = vu' > 0 \qquad \text{for} \quad v > 0 \qquad \text{and} \qquad (u, v) \in M_2^v. \tag{4.10}$$

To show (4.10), we write

$$M_2^\nu = \{(u, v) : u = U(\xi, \nu), v = V(\xi, \nu)\}$$

and prove that $V(\xi, 0) = 0$, $D_\nu V(\xi, 0) > 0$.

We have $M_2^0 = M_2$, hence

$$U(\xi, 0) = r_2(\xi), \qquad V(\xi, 0) = 0$$

and, consequently,

$$D_\xi U(\xi, 0) = r_2'(\xi) > 0. \tag{4.11}$$

Since M_2^ν is invariant for (4.2), we have

$$\nu D_\xi U(\xi, \nu) = V(\xi, \nu).$$

Differentiating with respect to ν at $\nu = 0$ and substituting from (4.11) we obtain

$$D_\nu V(\xi, 0) = r_2'(\xi) > 0. \quad \blacksquare$$

By (B3) of Section 2, for $\nu = 0$ the unstable manifold $W^u(M_1^\nu)$ intersects the line $W^s(M_2^\nu) \cap \{b = \Delta\} (= \{a = 0, \ b = \Delta\})$ transversally at $\xi = \mu_0$. Moreover, due to our orientation of a and b, from (B3) it follows that locally along the intersection curve a is an increasing function of ξ. By Proposition A1.3, this transversal intersection remains preserved for $\nu > 0$. In addition, the right hand boundary condition line $L_\nu(\mu)$ intersects the stable separatrix of the saddle point (0, 0) of (2.4_0) transversally. Therefore, by Proposition 3.1, the invariant manifold $\mathbf{\Gamma}$ (defined by (2.10)) uniformly C^2 $O(e^{-\rho/\varepsilon})$ approximates compact parts of $W^u(M_2^\nu)$. Consequently, for $\nu > 0$ small and ξ near γ_0 we have

$$\mathbf{\Gamma} \cap \{b = \Delta\} = \{(a, \Delta, \xi) : a = \sigma(\xi, \nu)\}$$

with σ being C^2,

$$\sigma(\xi, \nu) = 0 \qquad \text{for some} \quad \xi_0 = \xi_0(\nu) = \gamma_0 + O(\nu) \tag{4.12}$$

and

$$D_\xi \sigma(\xi_0, \nu) > 0.$$

Occasionally we will omit ν as an argument of σ.

A solution $u(\xi)$ of (4.2) which enters the box $|a| \leqslant \Delta$, $|b| \leqslant \Delta$ for some $\xi = \zeta$ through the plane $b = \Delta$ and stays there for $\xi \in [\zeta, \mu]$ satisfies the left hand boundary condition (4.3) if and only if it locally near M_2 transforms to a solution $(a(t), b(t), \xi(t))$ of (4.4)–(4.6) such that

$$b(0) = \Delta, \qquad \xi(0) = \zeta, \tag{4.13}$$

$$(a(T), b(T)) \in L_\nu(\mu), \qquad \text{i.e.,} \quad a(T) = m\nu + nb(T) + \Omega(b(T)), \tag{4.14}$$

$$\xi(T) = \mu, \tag{4.15}$$

$$(u(\,.\,), u'(\,.\,)) \in \mathbf{\Gamma}, \qquad \text{i.e.,} \quad a(0) = \sigma(\zeta, \nu). \tag{4.16}$$

4.2. Proposition. *For fixed $\Delta > 0$ sufficiently small, there exist $\kappa > 0$, $\nu_0 > 0$ and a function $T_0\colon [0, \nu_0] \to (0, \infty)$ with $T_0 \to \infty$ for $\nu \to 0$ such that for $\nu \leqslant \nu_0$ and $\mu \in [\gamma_0 - \kappa, \gamma_0 + \kappa]$:*

(i) *The boundary value problem* (4.4)–(4.6), (4.13)–(4.16) *has no solution for $T \leqslant T_0$.*

(ii) *For $T > T_0$, the boundary value problem* (4.4)–(4.6), (4.13)–(4.16) *has a unique solution $\hat{a}(\zeta, \mu)$, $\hat{b}(\zeta, \mu)$, $\hat{\xi}(\zeta, \mu)$. The functions $\hat{a}(\zeta, \mu)(0)$, $\hat{b}(\zeta, \mu)(0)$ are C^2 and satisfy the estimates* (A2.8)–(A2.10).

(iii) *The solution of* (4.16) *is given by the bifurcation equation*

$$F(\zeta, \mu) = 0, \qquad F(\zeta, \mu) = \hat{a}(\zeta, \mu)(0) - \sigma(\zeta) \tag{4.17}$$

for ζ. It undergoes a generic fold bifurcation in the interval $[\gamma_0 - \kappa, \gamma_0 + \kappa]$: there is a $\mu_0 \in [\gamma_0 - \kappa, \gamma_0 + \kappa]$ which is a C^1 function of ν with uniformly bounded derivative such that for $\mu < \mu_0$, $\mu = \mu_0$, $\mu > \mu_0$ (4.17) *has* 0,1,2 *solutions ζ in $[\gamma - 2\kappa, \mu)$ respectively. The solution curve is C^2, its μ-component having a nondegenerate minimum μ_0.*

Proof. Take $\kappa > 0$, $\Delta > 0$, $\nu_0 > 0$ so small that

$$\underline{n} = \inf_{\substack{\gamma_0 - 2\kappa \leqslant \mu \leqslant \gamma_0 + 2\kappa \\ 0 \leqslant \nu \leqslant \nu_0}} n(\mu, \nu) > 0, \tag{4.18}$$

$$\underline{\sigma}_1 = \inf_{\substack{\gamma_0 - 2\kappa \leqslant \mu \leqslant \gamma_0 + 2\kappa \\ 0 \leqslant \nu \leqslant \nu_0}} D_\zeta \sigma(\zeta, \nu) > 0, \tag{4.19}$$

$$\sigma(\xi, \nu) < 0 \qquad \text{for} \quad \gamma_0 - 2 \leqslant \xi \leqslant \gamma_0 - \kappa, \qquad 0 \leqslant \nu \leqslant \nu_0, \tag{4.20}$$

$$\sigma(\xi, \nu) > 0 \qquad \text{for} \quad \gamma_0 + \kappa \leqslant \xi \leqslant \gamma_0 + 2\kappa, \qquad 0 \leqslant \nu \leqslant \nu_0, \tag{4.21}$$

and denote

$$\bar{\sigma}_1 = \sup_{\substack{\gamma_0 - 2\kappa \leqslant \zeta \leqslant \gamma_0 + 2\kappa \\ 0 \leqslant \nu \leqslant \nu_0}} D_\zeta \sigma(\zeta, \nu),$$

$$\bar{m} = \sup_{\substack{\gamma_0 - 2\kappa \leqslant \mu \leqslant \gamma_0 + 2\kappa \\ 0 \leqslant \nu \leqslant \nu_0,\ |b| \leqslant \Delta}} m(\mu, \nu, b).$$

Then, because of (4.8) and (4.12), we have

$$a(0) = O(v) + O(\kappa)$$

and, consequently, by (4.7), (4.8),

$$[O(\kappa) + O(v)]\, e^{\bar{\alpha} T} \geqslant \tfrac{1}{2} \underline{n}\, \Delta e^{-\bar{\beta} T} \tag{4.22}$$

which proves (i).

Conclusion (ii) is proved in Proposition A2.1. We now prove (iii). We show that if Δ, κ, v are chosen sufficiently small then for $0 < v \leqslant v_0$ and $\mu \in [\gamma_0 - \kappa, \gamma_0 + \kappa]$ there is a unique $\zeta = \psi(\mu) \in [\gamma_0 - 2\kappa, \mu)$ such that

$$D_\zeta F(\zeta, \mu) < 0 \qquad \text{for} \quad \psi(\mu) < \zeta \leqslant \mu - vT_0$$

while (4.23)

$$D_\zeta F(\zeta, \mu) > 0 \qquad \text{for} \quad \zeta < \psi(\mu)$$

and

$$D_\zeta^2 F(\zeta, \mu) > 0 \tag{4.24}$$

for $\zeta = \psi(\mu)$. Then, we show that for $v > 0$ sufficiently small

$$D_\mu F(\zeta, \mu) < 0 \qquad \text{for} \quad \zeta = \psi(\mu) \tag{4.25}$$

and

$$F(\psi(\gamma_0 - \kappa), \gamma_0 - \kappa) > 0, \tag{4.26}$$

while

$$F(\psi(\gamma_0 + \kappa), \gamma_0 + \kappa) < 0. \tag{4.27}$$

From (4.25)–(4.27) we conclude that there is a unique $\mu_0 \in [\gamma_0 - \kappa, \gamma_0 + \kappa]$ such that

$$F(\psi(\mu_0), \mu_0) = 0. \tag{4.28}$$

Because of (4.23) we have

$$F(\zeta, \mu) > 0 \tag{4.29}$$

for $\mu < \mu_0$, $\zeta \in [\gamma_0 - 2\kappa, \mu]$, i.e., (4.17) has no solution for $\mu < \mu_0$.

From (4.18), (4.20), (4.22), (4.26), and (4.27) it follows that, for $\mu \in (\mu_0, \gamma_0 + \kappa)$, F has precisely two zeros, both of them different from $\psi(\mu)$.

Since $D_\zeta F$ does not vanish at those zeros, by the implicit function theorem the curve $F=0$ is C^2 and parametrizable by μ for $\mu>\mu_0$.

This concludes the proof of (iii) provided we establish the existence of ψ and prove the estimates (4.26)–(4.29).

By Proposition A2.1 and Lemma 4.1, we have

$$r_1 \leqslant v^j D_\zeta^j \hat{a}(\zeta,\mu)(0) \leqslant R_1+R_2 \qquad \text{for} \quad j=0,1,2, \tag{4.30}$$

where

$$R_1 = R_1(\zeta,\mu) = Ce^{-(\alpha+\beta)((\mu-\zeta)/v)},$$
$$R_2 = Cve^{-\alpha((\mu-\zeta)/v)},$$
$$r_1 = ce^{-(\bar{\alpha}+\bar{\beta})((\mu-\zeta)/v)},$$

with $C>0$, $c>0$ for v sufficiently small and T_0 sufficiently large.

If both

$$R_1 < \tfrac{1}{2}\underline{\sigma}_1 v \tag{4.31}$$

and

$$R_2 < \tfrac{1}{2}\underline{\sigma}_1 v, \tag{4.32}$$

we have

$$D_\zeta F(\zeta,\mu) < 0. \tag{4.33}$$

The inequalities (4.31) and (4.32) are satisfied if

$$\zeta < \mu + \frac{v}{\alpha+\beta}\log\frac{\underline{\sigma}_1 v}{2C}, \tag{4.34}$$

$$\zeta < \mu + \frac{v}{\alpha}\log\frac{\underline{\sigma}_1}{2C}, \tag{4.35}$$

respectively. Obviously, for v sufficiently small (4.34) implies (4.35). Hence, for v sufficiently small (4.33) is satisfied provided

$$\zeta < \mu + vp_1(v),$$

where

$$p_1(v) = \frac{1}{\alpha+\beta}\log\frac{\underline{\sigma}_1 v}{2C}.$$

On the other hand, we have

$$D_\zeta F(\zeta, \mu) > 0,$$

provided

$$r_1 > \bar{\sigma}_1,$$

i.e.,

$$\zeta > \mu + \nu p_2(\nu),$$

where

$$p_2(\nu) = \frac{1}{\bar{\alpha} + \bar{\beta}} \log \frac{2\bar{\sigma}_1 \nu}{c}.$$

Note that the use of the estimates of Proposition A2.1 is justified since $p_1(\nu)$, $p_2(\nu) \to -\infty$ for $\nu \to 0$, so $p_1(\nu)$, $p_2(\nu) < -T_0$ for $\nu > 0$ sufficiently small.

We now prove that for κ sufficiently small and $0 < \nu \leqslant \nu_0$ sufficiently small (4.24) holds for all $\zeta \in [\mu + \nu p_1(\nu), \mu + \nu p_2(\nu)]$. Indeed, by (4.30), for $\nu > 0$ sufficiently small we have

$$D_\zeta^2 F(\zeta, \mu) \geqslant \nu^{-2} r_1 - \bar{\sigma}_2,$$

where

$$\bar{\sigma}_2 = \sup_{\substack{\zeta \in [\gamma_0 - 2\kappa, \gamma_0 + 2\kappa] \\ 0 \leqslant \nu \leqslant \nu_0}} D_\zeta^2 F(\zeta, \nu).$$

Since for ν sufficiently small (4.29) and (4.33) hold for $\zeta = \mu + \nu p_1(\nu)$, $\zeta = \mu + \nu p_2(\nu)$, respectively, and (4.24) holds for ζ inbetween, by the intermediate value and the implicit function theorems there is a unique $\zeta = \psi(\mu) \in [\mu + \nu p_1(\nu), \ \mu + \nu p_2(\nu)]$ for which $D_\zeta F(\zeta, \mu) = 0$ (and, thus, (4.23)) holds true, ψ being a C^2 function of μ. Since $p_2(\nu) \to -\infty$ for $\nu \to 0$, $\psi(\mu) < \mu - \nu T_0$ for $\nu \leqslant \nu_0$ sufficiently small.

We have

$$\begin{aligned} D_\mu F(\zeta, \mu)|_{\zeta = \psi(\mu)} &< -\nu^{-1}\, d_1 e^{-(\bar{\alpha} + \bar{\beta})((\mu - \psi(\mu))/\nu)} + d_2 e^{-\alpha((\mu - \psi(\mu))/\nu)} \\ &< -\nu^{-1} d_1 e^{-(\bar{\alpha} + \bar{\beta})\, p_1(\nu)} + d_2 e^{-\alpha p_2(\nu)} \\ &= -\frac{d_1 \underline{\sigma}_1}{2C} + \frac{2 d_2 \bar{\sigma}_1 \nu}{C} < 0 \end{aligned} \tag{4.36}$$

for $\nu \leqslant \nu_0$ sufficiently small, $\mu \in [\gamma_0 - \kappa, \gamma_0 + \kappa]$ and some $d_1, d_2 > 0$.

We now prove (4.26), (4.27). By (4.12),

$$\sigma(\psi(\gamma_0-\kappa))<\sigma(\gamma_0-\kappa)<-\underline{\sigma}_1(\kappa-O(\nu)); \tag{4.37}$$

from (4.12) and

$$\psi(\gamma_0+\kappa)\geqslant\gamma_0+\kappa+\nu p_1(\gamma_0+\kappa)\geqslant\gamma_0+\kappa-O(\nu\,|\log\nu|)$$

it follows

$$\sigma(\psi(\gamma_0+\kappa))>\underline{\sigma}_1(\kappa-O(\nu\,|\log\nu|)). \tag{4.37}$$

On the other hand, by (A2.8) for $j=0$ we have

$$\mu-\psi(\mu)\geqslant\nu p_2(\nu)\geqslant c\nu\,|\log\nu|$$

for some $c>0$ hence

$$|\hat{a}(\psi(\mu),\mu)(0)|=O(e^{c\log\nu})=O(\nu^c). \tag{4.39}$$

The estimates (5.26), (5.27) follow immediately from (4.37)–(4.39). ∎

For the purpose of the use of Proposition 4.2 in the proof of Theorem 1.1 we return to the original boundary value problem (1.1), (1.2). We have

4.3. Proposition. *For fixed $\Delta>0$ sufficiently small there are $\kappa>0$ and $\varepsilon>0$ such that for $0<\varepsilon\leqslant\varepsilon_0$ and $\mu\in[\gamma_0-\kappa,\gamma_0+\kappa]$ the solutions of* (1.1), (1.2) *which enter the box $|a|\leqslant\Delta$, $|b|\leqslant\Delta$ and stay there for x increasing undergo a generic fold bifurcation in the sense of Proposition* 4.2.

Proof. Having proved Proposition 4.2 all we have to do is to turn its bifurcation result from the case of μ varying and ν fixed to the case of μ varying and $\varepsilon=\nu/\mu$ fixed. To this end we only need to check that the line $\varepsilon=\text{const}$ crosses the bifurcation line $\mu=\mu_0(\nu)$ in the (μ,ν) plane transversally for ε small. From (4.36) it follows by the implicit function theorem that μ_0 is C^1 function of ν. Since $D_\zeta F(\mu,\zeta,\nu)=0$ at $\zeta=\psi(\mu,\nu)$, by differentiating (4.28) with respect to ν we obtain

$$D_\mu F D\mu_0+D_\nu F=0$$

at $\mu=\mu_0$, $\zeta=\psi(\mu_0)$, hence

$$D\mu_0=((D_\mu F)^{-1}\,D_\nu F)(\mu_0,\psi(\mu_0),\nu).$$

Since $\psi<p_2$, by (4.36) and (A2.10), we obtain

$$D\mu_0=O(D_\nu F(\mu_0,\psi(\mu_0),\nu))=O(D_\nu\hat{a}(\mu_0,\psi(\mu_0),\nu))+O(D_\nu\sigma(\psi(\mu_0),\nu)).$$

By (A2.10), $D_v\hat{a}$ is bounded for $v \to 0$ and so is $D_v\sigma$. Therefore, $D\mu_0$ is bounded for $v \to 0$. On the other hand, for the line $\varepsilon = \text{const}$ passing though the point (v, μ_0) we have

$$\left.\frac{\partial \mu}{\partial v}\right|_{\mu=\mu_0} = \frac{\mu_0}{v}.$$

Therefore, it cannot be tangent to the curve $\mu = \mu_0(v)$ at their intersection point for v sufficiently small. ▮

5. THE BIFURCATION AT $\mu = \gamma_M$

Unlike the bifurcation at $\mu = \gamma_0$, the investigation of the bifurcation at $\mu = \gamma_M$ does not rely on any global geometric principle like transversality. In some sense the bifurcation is local: by Proposition 3.1 it can be localized to the neighborhood of the point $\mu = \gamma_M$, $u = \hat{u}$, $v = 0$. Yet, a search for a suitable normal form which would facilitate its investigation was not successful. Therefore, a mixture of ad hoc arguments has been employed.

Recall the definitions (2.9) of the invariant manifold $\mathbf{\Gamma}$ and its sections $\Gamma(x)$, $x \geqslant 0$. By $\Gamma^+(x_0)$ we denote the part of $\Gamma(x_0)$ consisting of points of the solutions of (2.1) staying in the halfplane $v \geqslant 0$ for $0 \leqslant x \leqslant x_0$ and we write $\mathbf{\Gamma}^+ = \bigcup_{0 \leqslant x \leqslant 1} \Gamma^+(x)$. To indicate the dependence of $\mathbf{\Gamma}, \mathbf{\Gamma}^+$ and their x-sections on μ we will occasionally label them by the corresponding subscript.

Recall that for $\gamma_0 < \xi < \gamma_M$ we denote by $H(\xi)$ the $v \geqslant 0$ part of the homoclinic of the point $(r_1(\xi), 0)$ of (2.4_ξ). We precede the main result of this section by an auxiliary lemma related to Lemma 2.2.

5.1. LEMMA. *Let $\gamma_0 < \mu_1 < \gamma_M$, $\mu_2 > \gamma_M$ and let $0 < x_0 < 1$ be such that $\gamma_0 < x_0\mu_1 < x_0\mu_2 < \gamma_M$, $i = 1, 2$. Then, there is an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and all $\mu_1 \leqslant \mu \leqslant \mu_2$ one has*

$$\Gamma_\mu^+(x_0) = G_\mu(x_0) \cup Q_\mu(x_0)$$

with $G_\mu(x_0)$ compact and $Q_\mu(x_0)$ closed unbounded, $G_\mu(x_0)$ tending to $H(\mu x_0)$ for $\varepsilon \to 0$ uniformly in μ while

$$\inf\{u\colon (u, 0) \in Q_\mu(x_0)\} - \sup\{u\colon (u, v) \in H(\mu x_0)\} > d \tag{5.1}$$

for some $d > 0$ (Fig. 5).

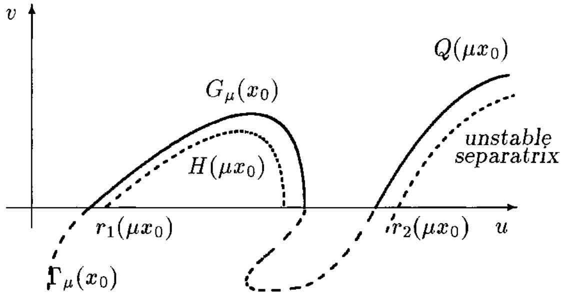

Fig. 5. The shape of $\Gamma_\mu(x_0)$.

Proof. For given $d>0$ denote

$$R=\{(u,v): u>\sup\{u: (u,v)\in H(\mu x_0)\}+d, v>0\}$$

and

$$G_\mu(x_0)=\Gamma^+(x_0)\cap R,$$

$$Q_\mu(x_0)=\Gamma^+(x_0)\backslash R.$$

Since $\Gamma^+(0)=\{u=0\}$ intersects $W^s(0,0)$ transversally, the part of $\Gamma^+(x_0)\cap R$ consisting of points the trajectories of which have not left R C^2 $O(\varepsilon)$-approximates $W^u(M_1^\varepsilon)\cap R\cap\{x=x_0\}=H(\mu x_0)$. Since $\dot{u}\geqslant 0$ along a trajectory in Γ^+, it cannot return to R after having left the latter. Consequently, $H(\mu x_0)$ is C^2 $O(\varepsilon)$-approximated by all $\Gamma^+(x_0)\cap R$. ▮

5.2. Corollary. *For $x\geqslant x_0$ one has*

$$\Gamma_\mu^+(x)=G_\mu(x)\cup Q_\mu(x).$$

Due to continuing uniform hyperbolicity of $r_2(\mu x)$ and Corollary 3.3 *applied to M_2, the inequality* (5.1) *extends to $x\geqslant x_0$.*

The main result of this section is

5.3. Proposition. *There exist $\mu_1<\gamma_M<\mu_2$ and $\varepsilon_0>0$ such that for all $0<\varepsilon\leqslant\varepsilon_0$ the following holds true*:

(i) *There is a $0<x_0<1$ such that $G_\mu(x)$ is compact connected and non empty for $\mu_1\leqslant\mu\leqslant\mu_2$, $x_0\leqslant x\leqslant l(\mu)$, where $l(\mu)=\sup\{x: G_\mu(y)\neq\varnothing$ for $x_0\leqslant y\leqslant x\}>x_0$ while $G_\mu(x)=\varnothing$ for $x>l(\mu)$.*

(ii) *A neighborhood of* $G_\mu(x)$ *in* $\Gamma(x)$ *is the graph of a function* $v = w(u, x, \mu)$, $\mu_1 \leqslant \mu \leqslant \mu_2$, $x_0 \leqslant x \leqslant l(\mu)$, $n_\mu^-(x) \leqslant u \leqslant n_\mu^+(x)$, *where* w *is* C^2 *in* u.

(iii) *The function* w *satisfies*

$$D_u^2 w < -c < 0 \tag{5.2}$$

for some $c > 0$.

(iv) *For the unique maximum* $m(\mu)$ *of* $w(\cdot, 1, \mu)$ *one has*

$$\frac{\partial}{\partial \mu} w(m(\mu), 1, \mu) < 0$$

while defined.

(v) *There is a unique* $\mu_M \in (\mu_1, \mu_2)$ *such that* $w(m(\mu_M), 1, \mu_M) = 0$; *one has* $w(m(\mu), 1, \mu) > 0$ *for* $\mu < \mu_M$, $G_\mu(1) = \varnothing$ *for* $\mu > \mu_M$ *and* $\mu_M \to \gamma_M$ *for* $\varepsilon \to 0$.

We prepare the proof of Proposition 5.3 by several lemmas.

Take $\mu_1 < \gamma_M < \mu_2$, $\gamma_0/\mu_1 < x_0 < \gamma_M/\mu_2$ and $u^- < u^+$ in such a way that

$$h''(u) < 0 \qquad \text{for} \quad u \in [u^-, u^+], \tag{5.3}$$

$$u^- < r_1(x_0\mu) < \max\{u\colon (u, v) \in H(x_0\mu)\} < u^+ \tag{5.4}$$

for $\mu_1 \leqslant \mu \leqslant \mu_2$. Note that then we also have

$$h(u^\pm) - \mu x_0 < 0.$$

We first establish the existence of w claimed in (ii) for $x = x_0$. To this end we consider the energy function

$$E_x(u, v) = \frac{v^2}{2} - \int_0^u h(y)\, dy + \mu x u.$$

In the lemma below we show that E_x increases with u over $G_\mu(x)$. To this end denote $(U(x, s), V(x, s))$ the values at x of the solution of (2.1) through the point $x = u = 0$, $v = s$.

5.4. Lemma.

$$D_s E_x(U(x, s), V(x, s)) > 0 \tag{5.5}$$

for all $0 < x < l(\mu)$, $\mu_1 \leqslant \mu \leqslant \mu_2$ *and all* $(U(x, s), V(x, s)) \in G_\mu(x)$.

Proof. We have $E_x(U(0, s), V(0, s)) = E_x(0, s) = s^2/2$. Hence, (5.5) holds for $x = 0$, $s > 0$. Further, we have

$$\begin{aligned} D_x D_s E_x(U(x, s), V(x, s)) &= D_x(VV_s - h(U)\, U_s + \mu x U_s) \\ &= V_x V_s + VV_{xs} - h'(U)\, U_x U_s - h(U)\, U_{sx} + \mu U_s + \mu x U_{sx} \\ &= \frac{1}{\varepsilon}\,[(h(U) - \mu x)\, V_s + Vh'(U)\, U_s - h'(U)\, VU_s \\ &\qquad - h(U)\, V_s + \mu U_s + \mu x V_s] \\ &= \frac{1}{\varepsilon}\,\mu\, D_s U(s, x). \end{aligned} \tag{5.6}$$

Since $D_s V(s, 0) = 1 > 0$, and $\varepsilon D_x D_s U(s, x) = D_s V(s, x)$, we have $D_s V(s, x) > 0$ and, consequently,

$$D_s U(s, x) > 0 \tag{5.7}$$

over $G_\mu(x)$ for $x > 0$ small. From (5.6) and (5.7) it follows that (5.9) holds over $G_\mu(x)$ for $x > 0$ sufficiently small.

Denote

$$x^* = \sup\{x\colon D_s E_x(U(s, x), V(s, x)) > 0 \text{ for } (U(s, x), V(s, x)) \in G_\mu(x)\}.$$

By contradiction we prove that (5.12) holds over $G_\mu(x)$ for $0 < x < x^*$.

Assume that there is an $0 < x_1 < x^*$ such that $D_s U(s, x) > 0$ over $G_\mu(x)$ for all $0 < x < x_1$ but there is an s_1 such that $(U(s_1, x_1), V(s_1, x_1)) \in G_\mu(x_1)$ and $D_s U(s_1, x_1) = 0$. Since $x_1 < x^*$ and, thus, $D_s E_x(U(s_1, x_1), V(s_1, x_1)) > 0$, we have necessarily $D_s V(s_1, x_1) > 0$. Consequently,

$$D_s V(s_1, x)/D_s U(s_1, x) \to \infty \qquad \text{for} \quad x \to x_1. \tag{5.8}$$

However, we have

$$\frac{d}{ds}\left(\frac{V_s}{U_s}\right) = V_{sx}/U_s - U_{sx} V_s/U_s^2 = \frac{1}{\varepsilon}\left[h'(U) - \frac{V_s^2}{U_s^2}\right] \leqslant \frac{1}{\varepsilon}\, h'(U).$$

Since $h'(U(s_1, x))$ is bounded for $x \leqslant x_1$, (5.8) is impossible. Now, if $x^* < l(\mu)$ we would necessarily have $D_s E_x(U(s, x^*), V(s, x^*)) = 0$ for some $(U(s, x^*), V(s, x^*)) \in G_\mu(x^*)$ while $D_s E_x(U(s, x), V(s, x)) > 0$ for $x < x^*$. Since (5.6) holds for $x < x^*$, this is impossible. ∎

5.5. COROLLARY. *There is an* $\varepsilon_0 > 0$ *such that, for* $0 < \varepsilon \leqslant \varepsilon_0$, (ii) *of Proposition* 5.3 *holds true for* $x = x_0$.

Indeed, the tangents to $H(\mu x_0)$ have a nonvanishing u-component at all points except the right intersection point R with the u-axis. This property is locally near R shared by the neighboring trajectories of $H(\mu x_0)$. From Corollary 3.2 it follows that $G_\mu(x)$ is the graph of a C^2 function except of, possibly, a neighborhood of R. The level lines of E_{x_0} are the trajectories of (2.3) for $\varepsilon = 0$. Thus, Lemma 5.3 implies that with s increasing, i.e., concurrently with forward movement along the trajectories, $G_\mu(x)$ intersects them inside out. Therefore, the u-components of the tangents of $G_\mu(x)$ cannot vanish at any of its points even in the neighborhood of R. This property, which is equivalent to $G_\mu(x)$ being a C^2 $u \mapsto v$ graph, obviously extends to a neighborhood of $G_\mu(x)$ in $\Gamma(x)$.

For the following lemma we note that, for $\mu_1 \leqslant \xi \leqslant \mu_2$, $H(\xi)$ is the graph of a function

$$v = g_\xi(u).$$

As $H(\xi)$ is a part of the unstable manifold of $(r_1(\xi), 0)$, g_ξ is C^2 in a neighborhood of $r_1(\xi)$.

5.6. LEMMA.

$$g''_\xi(r_1(\xi)) = (6h'(r_1(\xi))^{-1/2}\, h''(r_1(\xi)) \tag{5.9}$$

for $\mu_1 x_0 \leqslant \xi \leqslant \mu_2 x_0$.

Proof. $H(\xi)$ is tangent to the unstable eigenvector $(1, h'(r_1(\xi))^{1/2})$ of the saddle $(r_1(\xi), 0)$ of (2.4_ξ), hence

$$g_\xi(u) = h'(r_1(\xi))^{1/2}\,(u - r_1(\xi)) + \lambda(u - r_1(\xi))^2 + o(u - r_1(\xi))^2.$$

To compute λ, we evaluate the quadratic term of the invariance condition

$$Dg_\xi(u)\; g_\xi(u) = h(u) - \xi$$

of (2.4_ξ). We obtain

$$1/2h''(r_1(\xi)) = 3h'(r_1(\xi))^{1/2}\,\lambda,$$

which is (5.9). ∎

5.7. LEMMA. *For $\mu_1 \leqslant \mu \leqslant \mu_2$ and $0 < \varepsilon < \varepsilon_0$ sufficiently small, the map w of Proposition* 5.3(ii) *extends to all $x \leqslant l(\mu)$ and satisfies* (5.2) *for a suitable κ. Further, w satisfies the differential equation*

$$w \frac{\partial w}{\partial u} + \varepsilon \frac{\partial w}{\partial x} = h(u) - \mu x \tag{5.10}$$

while defined.

Proof. From Corollary 5.5 it follows by continuity that, for all $\mu_1 \leqslant \mu \leqslant \mu_2$, w extends to $x \in [x_0, x_0 + 2\delta]$ for some $\delta > 0$. The equation (5.10) is a simple consequence of invariance of $\mathbf{\Gamma}$. Indeed, if $(u(x), v(x))$ is a solution of (2.1) and $(u(x), v(x)) \in G_\mu(x)$, we have $v(x+h) = w(u(x+h), x+h)$ for $|h|$ small. Differentiating with respect to h we obtain

$$w_u u' + w_x = v'.$$

Substituting for u', v' from (2.1) and multiplying by ε we obtain (5.10).

By Lemma 5.6 and Corollary 3.3, there is a neighborhood U of $(r_1(\mu x_0), 0)$ such that

$$D_u^2 w(u, x_0, \mu) < -\kappa_1 < 0$$

for some $\kappa_1 > 0$, provided $0 < \varepsilon < \varepsilon_0$, for some ε_0 sufficiently small and $\mu_1 \leqslant \mu \leqslant \mu_2$. From Remark 2.3 it follows that if ε_0 is sufficiently small, $0 < \varepsilon < \varepsilon_0$, and $(u(x), v(x))$ is a solution of (2.1) such that $(u(x), v(x)) \in G_\mu(x_0 + \delta)$ then $(u(x_0), v(x_0)) \in G_\mu(x_0) \cap U$. Therefore, to complete the proof of the lemma (with x_0 possibly replaced by $x_0 + \delta$) it remains to be shown that if $(u(x), v(x))$ is a trajectory of (2.1) such that $(u(x_0), v(x_0)) \in U \cap G_\mu(x_0)$ and $v(y) = w(u(y))$ is defined for $x_0 \leqslant y \leqslant x$ then w satisfies (5.2) at $u = u(x)$, x, μ with κ depending on c_1 only.

To this end we first estimate w_u. Differentiating (5.10) along the solution $(u(x), v(x))$ we obtain

$$\frac{d}{dx} w_u = \frac{1}{\varepsilon} w_{uu} w + w_{ux}. \tag{5.11}$$

Differentiating (5.10) with respect to u we obtain

$$w_{uu} w + w_u^2 + \varepsilon w_{ux} = h'(u). \tag{5.12}$$

Substituting for $w_{uu} w + \varepsilon w_{ux}$ from (5.12) into (5.11) we obtain

$$\frac{d}{dx} w_u = \frac{1}{\varepsilon} [h'(u) - w_u^2] \leqslant \frac{1}{\varepsilon} h'(u).$$

Hence, w_u is bounded for $x \geqslant x_0$ along the solutions $(u(x), v(x))$ satisfying $u(x_0),\ v(x_0) \in G_\mu(x_0) \cap U$ from above by

$$b := \max_{u \in U \cap G_\mu(x_0)} w_u(u, x_0, \mu) + \frac{1}{\varepsilon}(1 - x_0) \max_{0 \leqslant u \leqslant \hat{u}} h'(u).$$

Along $(u(x), v(x))$,we have

$$\frac{d}{dx} w_{uu} = \frac{1}{\varepsilon} w_{uuu} w + w_{uux} \tag{5.13}$$

and, differentiating (5.12) with respect to u we obtain

$$w_{uuu} w + 3w_u w_{uu} + \varepsilon w_{uux} = h''(u). \tag{5.14}$$

Substituting from (5.14) into (5.13) we obtain

$$\frac{d}{dx} w_{uu} = \frac{1}{\varepsilon}[h''(u) - 3w_u w_{uu}].$$

Hence, we have

$$\begin{aligned} w_{uu}(x, u(x)) &= w_{uu}(x_0, u(x_0))\, e^{-(3/\varepsilon)\int_{x_0}^{x} w_u(s,\, u(s))\, ds} \\ &\quad + \frac{1}{\varepsilon}\int_{x_0}^{x} e^{-(3/\varepsilon)\int_{s}^{x} w_u(s,\, u(s))\, ds} h''(u(s))\, ds \\ &\leqslant -\kappa_1 e^{-(3b/\varepsilon)(x - x_0)} \\ &\quad + \frac{1}{3b} \sup_{\substack{n^-(x,\mu) < u < n^+(x,\mu) \\ x_0 \leqslant x \leqslant 1}} h''(u)[1 - e^{-3b(x - x_0)}]. \end{aligned} \tag{5.15}$$

The right-hand side of (5.15) obviously has a negative supremum for $x \leqslant l(\mu)$ provided $u(x)$ stays in an interval where $h''(u)$ has a negative supremum. Because of (5.4) and Lemma 5.1, to prove the latter and, thus, complete the proof of the lemma, it suffices to prove that $\pm n^{\pm}$ decreases if $n^{\pm}$ is outside the interval of the u-axis bounded by points of its intersection with $H(\mu x)$, i.e.,

$$\pm \frac{d}{dx} n^{\pm}(x, \mu) < 0 \qquad \text{if} \quad h(n^{\pm}(x, \mu)) - \mu x < 0.$$

We have

$$w(n^{\pm}(x, \mu), x) = 0. \tag{5.16}$$

Differentiating with respect to x we obtain

$$w_u(n^\pm, x)\,\dot{n}^\pm = -w_x(n^\pm, x).$$

Substituting for w_x from (5.11) and taking (5.16) into account we obtain

$$\frac{d}{dx} n^\pm = -\frac{1}{\varepsilon w_u(n^\pm, x)}\,[h(u^\pm) - \mu x].$$

Because of $\pm w_u(n^\pm) < 0$ and (5.3), this completes the proof of the lemma. ▮

Proof of Proposition 5.3. Conclusion (i) is an immediate consequence of Lemma 5.1, conclusions (ii) and (iii) follow from Lemma 5.7. It remains to prove (iv) and (v).

Since $\partial/\partial u\; w(u, 1, \mu)\,|_{u=m(\mu)} = 0$, we have

$$\frac{\partial}{\partial \mu} w(m(\mu), 1, \mu) = \frac{\partial}{\partial \mu} w(u, 1, \mu)\,|_{u=m(\mu)}.$$

By differentiating the Eq. (2.1) with respect to μ we conclude that

$$\frac{\partial w}{\partial \mu}(u, 1, \mu)\,|_{u=m(\mu)} = q(1),$$

where $(p(x), q(x))$ solves the differential equation

$$\begin{aligned} \varepsilon p' &= q, \\ \varepsilon q' &= h'(u_m(x))\, p - \mu x, \\ p(0) &= q(0) = 0, \end{aligned} \tag{5.17}$$

and $u_m(x)$ is the solution of (1.1) satisfying $u(0) = 0$, $u_m(1) = m(\mu)$. Expressing (p, q) in polar coordinates, $p = \rho \cos \vartheta$, $q = \rho \sin \vartheta$ (at points where $(p, q) \neq 0$), from (5.17) we conclude that ϑ satisfies the equation

$$\varepsilon \vartheta' = h'(u_m(x)) \cos^2 \vartheta - \sin^2 \vartheta - \frac{\mu x}{\rho} \cos \vartheta$$

for those x for which $\rho(x) \neq 0$.

We prove that

$$\vartheta(x) \in (\pi, 3\pi/2) \qquad \text{for} \quad \rho(x) > 0, \qquad 0 < x \leqslant 1. \tag{5.18}$$

Note that (5.18) proves (5.16) and, hence, completes the proof of Proposition 5.3.

To prove (5.18) we compare $\vartheta(x)$ to the argument $\varphi(x)$ of the tangent to $G_\mu(x)$ at $u_m(x)$. The evolution of the vector $(y(x), z(x))$ generating this tangent (oriented downward at $x=0$) is governed by the linearization of the first two equations of (2.1) along $u_m(x)$, i.e., the equation

$$\dot{y} = z,$$
$$\dot{z} = h'(u_m(x))\,y,$$
$$y(0) = 0, \qquad z(0) = -1.$$

Hence, φ satisfies

$$\varepsilon\varphi' = h'(u_m(x))\cos^2\varphi - \sin^2\varphi, \qquad \varphi(0) = \frac{3\pi}{2}. \tag{5.19}$$

We prove

5.8. LEMMA. *One has*

$$\varphi(x) \in (\pi, 3\pi/2) \tag{5.20}$$

for $0 < x \leqslant 1$ *and*

$$\varphi(1) = \pi.$$

Proof. Since the tangent to $G_\mu(1)$ is horizontal at $u = m(\mu)$, we have

$$\varphi(1) \in \{0, \pi\}. \tag{5.21}$$

If $\varepsilon > 0$ is sufficiently small, by Remark 2.3, $u_m(x)$ is so close to $r_1(\mu x)$ for $0 \leqslant x < x_0$ that $h'(u_m(x)) > 0$. From (5.19) it follows $\varphi' > 0$ if $\varphi = \pi$ while $\varphi' < 0$ if $\varphi = 3\pi/2$, $0 \leqslant x \leqslant x_0$. Hence, (5.20) holds for $0 < x < x_0$.

By Lemma 5.6, $G_\mu(x)$ is a graph of a concave function $u \mapsto v$ for $x_0 \leqslant x \leqslant 1$. Therefore, the tangent to $G_\mu(x)$ is never vertical for $x_0 \leqslant x \leqslant 1$, i.e., $\varphi(x) \notin \{3\pi/2, \pi/2\}$ for $x_0 \leqslant x \leqslant 1$. Consequently, $\varphi(x) \in (\pi/2, 3\pi/2)$ for $0 < x \leqslant 1$ which, together with (5.21), implies $\varphi(1) - \pi$. If

$$\varphi(x_1) < \pi \qquad \text{for some} \quad x_1 < 1, \tag{5.22}$$

there is a $x_2 < x_1$ such that $\varphi(x_2) = \pi$, $\varphi'(x_2) \leqslant 0$. Since $u_m(x)$ is increasing, $h'(u_m(x))$ is decreasing. Hence, $\varphi'(x) < \varphi'(x_1) < \varphi'(x_2) = 0$ for $x \geqslant x_1$. This makes (5.22) impossible and proves the lemma. ∎

Returning to the proof of Proposition 5.3 we note that $p'(0) = p''(0) = q(0) = q'(0) = 0$ while $q''(0) = p'''(0) = -\mu/\varepsilon$. Therefore, we have $p(x) < 0$,

$q(x)<0$ and $p(x)=o(q(x))$, hence $3\pi/2>\vartheta(x)=3\pi/2-o(x)$ for $x>0$ small. Since $\varphi'(0)=-1/\varepsilon$ and $\varphi(0)=3\pi/2$, we have

$$\varphi(x)<\vartheta(x)<3\pi/2 \tag{5.23}$$

for $x>0$ small.

The inequality (5.23) remains valid as long as $\vartheta(x)$ is defined. Indeed, $\vartheta'(x)<0$ if $\vartheta=3\pi/2$ and $\vartheta'>\varphi'$ if $\vartheta=\varphi$. from Lemma 5.8 it follows that $\vartheta(x)\in(\pi, 3\pi/2)$ for all $0<x\leqslant 1$ provided we prove that $\vartheta(x)$ is defined on this interval.

Suppose the contrary. Then, there is an $x_1\in(0,1]$ such that $\vartheta(x)$ is defined for $x<x_1$ but $\vartheta(x_1)$ is not, i.e. $p(x_1)=q(x_1)=0$. Since $\vartheta(x)\in(\pi, 3\pi/2)$ for $x<x_1$ we have $p'(x_1)\geqslant 0$, $q'(x_1)\geqslant 0$. On the other hand, from (5.17) it follows $q'(x_1)=-\mu x_1<0$. This contradiction completes the proof of (iv).

To prove (v) we first note that $G_{\mu_1}(1)\neq\varnothing$ by Corollary 3.2 while $G_{\mu_2}(1)=\varnothing$ by Lemma 2.2, provided ε_0 was chosen sufficiently small. From (iv) it follows by the implicit function theorem that $w(m(\mu), 1, \mu)$ is continuous while defined. Therefore, the existence of μ_M follows from the intermediate value theorem, its uniqueness from (iv). Finally, $\mu_M\to\gamma_M$ because of Proposition 2.1 and Lemma 2.2. ∎

Observing that $n_\mu^\pm(1)$ are the only zeros of $w(u, 1, \mu)$, from (iii) and (v) of Proposition 5.3 we obtain

5.9. Corollary. *The zeros of the equation $w(u, 1, \mu)=0$ undergo a generic fold bifurcation at $\mu=\mu_M$.*

Note that the zeros of $w(\cdot, 1, \mu)$ represent the solutions of (1.1), (1.2).

6. PROOF OF THEOREM 1.1

In this section we synthesize the results of Sections 2–5 to prove Theorem 1.1.

First, we choose $\Delta>0$, $\kappa>0$, $\mu_1<\gamma_M<\mu_2$ such that Propositions 3.3 and 5.3 hold true. Then, we take $\delta\leqslant\frac{1}{2}\min\{\kappa, \gamma_m-\mu_1, \mu_2-\gamma_M\}$, $d>0$ and $\varepsilon_0>0$ so small that Proposition 2.1 holds true and the neighborhoods $D_i=\bigcup_{0\leqslant 1} D_i(x)$, $i=1, 2$, $D_i(x)=\{(u, v): |u-r_1(\mu x)|\leqslant d,\ |v|\leqslant d\}$ can be coordinated by Fenichel coordinates for $0\leqslant\mu\leqslant\gamma_M$, $\gamma_0\leqslant\mu\leqslant\bar{\mu}$, respectively. We split the proof into five cases:

(i) $0 \leqslant \mu < \gamma_0 - 2\delta$,

(ii) $\gamma_0 - 2\delta \leqslant \mu \leqslant \gamma_0 + 2\delta$,

(iii) $\gamma_0 + 2\delta < \mu < \gamma_M + 2\delta$,

(iv) $\gamma_M - 2\delta \leqslant \mu \leqslant \gamma_M + 2\delta$,

(v) $\gamma_M + 2\delta < \mu < \bar{\mu}$.

Below, $(u(x), v(x))$ stands for the (u, v)-component of a trajectory of (2.3), representing a solution of (1.1), (1.2).

Case (i). Denote $R = \{(u, v): u \leqslant \max_{\mu \leqslant \bar{\mu}} r_2(\mu)\}$. Since $v' > 0$ for (u, v) outside R, $(u(x), v(x)) \in \mathbf{\Gamma} \cap R$ for all $0 \leqslant x \leqslant 1$. The line $\Gamma(0)(=\{u=0\})$ intersects $W^s(0, 0)$ transversally at $u = v = 0$. Therefore, by Corollary 3.2, $\Gamma^+(1) \cap R$ C^2 $O(\varepsilon)$-approximates $W^u(M_1^\varepsilon) \cap \{x = 1\} = W^u(r_1(\mu), 0)$. Since the latter has a uniques transversal intersection with the right-hand boundary value line $v = 1$, so has $\Gamma^+(1)$.

Case (ii). Because of the choice of δ, $W^u((r_2(\mu x), 0)$ enters D_2 for some $x \in [1 - \delta, 1]$. By the approximation argument of Case (i), $\Gamma^+(x)$ has to enter D_2 as well, Since the points $(u, v, x) \in \mathbf{\Gamma}^+ \backslash (D_1 \cup D_2)$ satisfy $v > \sigma$ for some $\sigma > 0$. $(u(x), v(x), x)$ either stays in D_1 for all x or enters D_2. As in Case (i), there is exactly one solution in D_1,namely u_1.

Applying the argument of (i) to D_2 instead of D_1 we conclude that once $(u(x), v(x), x)$ enters D_2 it has to stay there. Therefore, there are no more solutions except of u_1 and those provided by Proposition 4.3. The latter represent u_2, u_3; Proposition 4.3 establishes their generic fold bifurcation.

Case (iii). By Corollary 3.2, $\mathbf{\Gamma}^+$ C^2 $O(\varepsilon)$-approximates the compact subset $S = W^u(M_1^\varepsilon) \cap [R \backslash (D_1 \cup D_2)] \cap \{v = 0\}$ of $W^u(M_1^\varepsilon)$. From Remark 3.3 it follows that, since $W^u(M_1^\varepsilon)$ enters D_2, for $\mu x \geqslant \gamma_0 - \delta$ the trajectories in $\mathbf{\Gamma}^+ \backslash D_1$ can leave the $O(\varepsilon)$-neighborhood of S into D_2 only. For ε sufficiently small, $W^u(M_1^\varepsilon)$ intersects $W^s(M_2^\varepsilon) \cap \{b = \Delta\}(=\{a = 0, b = \Delta\}$ transversally in a unique point (a, b being the Fenichel coordinates in D_2). By Corollary 3.2, so does $\mathbf{\Gamma}^+$. Applying Corollary 3.2 again with $\Sigma^\varepsilon = \mathbf{\Gamma}^+ \cap \{b = \Delta\}$ we conclude that $\Gamma^+(1)$ C^2 $O(\varepsilon)$-approximates $W^u(M_2^\varepsilon)$ in D_2 and, therefore, has a unique intersection with $\{v = 0\}$ giving the solution u_2.

The subset of $\Gamma^+(1)$ of points the trajectories of which do not enter D_2 C^2 $O(\varepsilon)$-approximates $S \cap \{x = 1\} = H(\mu)$. Therefore, it intersects $\{v = 0\}$ in two points giving u_1 and u_3.

Case (iv). As in Case (iii), the solution u_2 is given by the unique intersection of $\Gamma^+(1) \cap D_2(\subset Q_\mu(1)$ of Lemma 5.1) with the line $v = 0$. The solutions u_1, u_3 and their generic bifurcation follow by Proposition 5.3.

Case (v). Recall the definition of G_μ in Lemma 5.1. Since $G_\mu(1)=\varnothing$ for $\mu>\mu_M+\delta$ and ε_0 sufficiently small, $\Gamma^+(1)\cap S=\varnothing$ for such ε. Therefore, only the solution u_2 remains, whose existence and uniqueness follows as in Case (iii). ∎

APPENDIX 1

In this Appendix we summarize results of geometric singular perturbation theory (GSP) relevant to our problem. In order to keep presentation simple we restrict ourselves to the case of two-dimensional fast variable and one-dimensional slow variable the flow of which is parallel. That is, we consider systems of form

$$\varepsilon y' = f(x, y, \varepsilon) \tag{A1.1}$$

or their fast time version

$$\dot{y} = f(x, y, \varepsilon) \tag{A1.2}$$

$$\dot{x} = \varepsilon \tag{A1.3}$$

with $y \in \mathbb{R}^2$ and $f \in C^4$.

For proofs and GSP in a more general context the reader is referred to [5, 6, 9, 15, 16].

We suppose that f satisfies the assumptions of Section 3, i.e., for $0 \leqslant x \leqslant X$ the equation

$$f(x, y, 0) = 0 \tag{A1.4}$$

admits a continuous curve M: $y=r(x)$ of solutions such that $D_y f(x, r(x), 0)$ has a pair of real nonzero eigenvalues of different sign. Equivalently, this means that for each fixed $x \in [0, X]$ the point $r(x)$ is a saddle for the equation (A1.2) with $\varepsilon=0$. Each of those points is in the intersection of two 1-dimensional C^2 invariant manifolds: the unstable manifold $W^u(r(x))$ and the stable manifold $W^s(r(x))$ depending on x in a C^4 way. Those are, by definition, the sets of points the trajectories of which tend to the saddle for $t \to -\infty$ and $t \to +\infty$ respectively. The unstable manifold of M is defined by

$$W^u(M_1^\varepsilon) = \bigcup_x \{(x, y) \colon y \in W^u(r(x))\}. \tag{A1.5}$$

The stable manifold of M is defined similarly. Both manifolds are C^4.

A1.1. PROPOSITION. *There exists a 2-dimensional* C^4 *submanifold* $\mathbf{M}$ *of the* (x, y, ε)*-space meeting the hyperplanes* $\varepsilon = const$ *transversally such that, for some* $\varepsilon_0 > 0$, $0 \leqslant \varepsilon \leqslant \varepsilon_0$, $M^\varepsilon = \mathbf{M} \cap \{\varepsilon = const\}$ *are* C^4 *isotopic to* M *locally invariant manifolds of* (A1.2), (A1.3) *and* $M_0 = M$. *There exist* 3*-dimensional* C^4 *submanifolds* $W^u(\mathbf{M})$, $W^s(\mathbf{M})$ *of the* (x, y, ε)*-space meeting the hyperplanes* $\varepsilon = const$ *transversally such that* $W^j(\mathbf{M}) \cap \{\varepsilon = 0\} = W^j(M)$, *and* $W^j(M^\varepsilon) = W^j(\mathbf{M}) \cap \{\varepsilon = const\}$ *is a locally invariant manifold of* (A1.2), (A1.3) *containing* M^ε *for* $j = u, s$. *The trajectories in* $W^u(M^\varepsilon)$ *resp.* $W^s(M^\varepsilon)$ *approach* M^ε *uniformly exponentially for* t *decreasing resp. increasing.*

A1.2. PROPOSITION (Fenichel coordinates). *The system* (A1.2), (A1.3) *can be locally near* M^ε *be transformed to the form*

$$\begin{aligned} \dot{a} &= A(a, b, x, \varepsilon)\, a \\ \dot{b} &= B(a, b, x, \varepsilon)\, b \\ \dot{x} &= \varepsilon \end{aligned} \tag{A.16}$$

with $A(a, b, x, \varepsilon) > \alpha > 0$, $B(a, b, x, \varepsilon) < -\beta < 0$ *being* C^2, *by a* C^3 *transformation depending continuously on* ε *in the* C^3 *topology.*

In (A1.6), we have $M^\varepsilon = \{a = b = 0\}$, $W^u_{loc}(M^\varepsilon) = \{b = 0\}$, $W^s(M^\varepsilon) = \{a = 0\}$.

A1.3. PROPOSITION. *There exists a family of* C^4 *maps* $\Psi^\varepsilon \colon [0, X] \times \mathbb{R} \to \mathbb{R}^3$ *such that* Ψ^ε *maps* C^4 *diffeomorphically* $[0, X] \times \mathbb{R}$ *onto* $W^u(M^\varepsilon)$, $\Psi^\varepsilon([0, X] \times \{0\} = M^\varepsilon\}$ *and* $\Psi^\varepsilon \to \Psi^0$ *uniformly* C^4 *over compacts. A similar conclusion holds for* $W^s(\mathbf{M})$.

Locally, Proposition A1.3 is a consequence of A1.2. To globalize Ψ^ε one can proceed as in [1, 27.4].

A1.4. COROLLARY. *Let* $K \subset [0, X] \times \mathbb{R}$ *be compact and let* $I \subset \mathbb{R}$ *be an open interval containing* 0 *in its interior. Then* $\Psi^\varepsilon|_K$ *can be extended to a diffeomorphism of* $K \times I$ *to a tubular neighborhood* W^ε *of* $\Psi^\varepsilon(K)$ *with* $\Psi^\varepsilon(K) = \Psi^\varepsilon(K \times \{0\})$ *and the convergence property of Proposition* A1.3 *preserved. A similar conclusion holds for* $W^s(\mathbf{M})$.

To extend Ψ^ε one just takes a transversal bundle of $W^u(\mathbf{M}^\varepsilon)$ over $\Psi^\varepsilon(K)$ and maps the sets $\{z\} \times I$, $z \in K$ onto the fibers of the bundle.

APPENDIX 2

In this Appendix we compute the leading terms of the function $\hat{a}(\mu, \xi)$, entering the bifurcation equation (4.17). To this end we employ the techniques of [9, Section 4] with some modifications.

We will consider system (4.4)–(4.6) in the domain

$$\gamma_0 - 2\kappa \leqslant \xi \leqslant \gamma_0 + 2\kappa, \qquad |a|, |b| \leqslant \Delta, \qquad 0 \leqslant v \leqslant v_0, \tag{A2.1}$$

for certain κ, Δ, v_0 sufficiently small. In order to avoid technicalities we extend A, B to $\mathbb{R}^4$ in such a way that they remain C^2-bounded and satisfy

$$\bar{\alpha} > A > \alpha > 0, \qquad -\bar{\beta} < B < -\beta < 0. \tag{A2.2}$$

To simplify the formulas we will freely drop arguments of functions the values of which will be obvious from the context. Estimates including the symbols O, o will always be understood to hold uniformly for a, b, ξ, v satisfying (A2.1).

Denote

$$A_0(t) = A_0(t, \zeta, v) := A(0, q(t), \zeta + vt, v),$$

$$B_0(t) = B_0(t, \zeta, v) := B(0, q(t), \zeta + vt, v),$$

$$\mathscr{A}(t, \tau) := \int_\tau^t A_0(s)\, ds, \qquad \mathscr{B}(t, \tau) = \int_\tau^t B_0(s)\, ds,$$

$$\tilde{B}(t) = B_0(t) + D_b B(0, q(t), \zeta + vt)\, q(t),$$

$$\tilde{\mathscr{B}}(t, \tau) = \int_\tau^t \tilde{B}(s)\, ds,$$

where $q(t) = q(t, \zeta, v)$ is the solution of (4.5) with $a = 0$ and $\xi = \zeta + vt$ satisfying $q(0) = \Delta$. Note that

$$q(t) = \Delta e^{\mathscr{B}(t, 0)}. \tag{A2.3}$$

The result of this Appendix is the following:

A2.1. Proposition. *For fixed κ, Δ sufficiently small, a, b, v satisfying (A2.1), $\gamma_0 - \kappa \leqslant \mu \leqslant \gamma_0 + \kappa$, $\gamma_0 - 2\kappa \leqslant \zeta < \mu$ there exist $T_0 > 0$, $v_0 > 0$ such that for $T \geqslant T_0$ and $v \leqslant v_0$ the boundary value problem*

$$\dot{a} = A(a, b, \zeta + \nu t, \nu)a \tag{A2.4}$$

$$\dot{b} = B(a, b, \zeta + \nu t, \nu)b \tag{A2.5}$$

$$b(0) = \Delta, \qquad a(T) \in L_\nu(\mu) \tag{A2.6}$$

$$\nu T = \mu - \zeta \tag{A2.7}$$

(L *given by* (4.7)) *has a unique solution* $\hat{a}(\mu, \zeta, \nu)$, $\hat{b}(\mu, \zeta, \nu)$. *This solution is* C^2 *in* μ, ζ, ν *and satisfies the following estimates*:

$$D^j_\zeta \hat{a}(\mu, \zeta, \nu)(0) = \nu^{-j}\{[(A_0(0) - B_0(T))^j + O(e^{-\delta T}) + O(\Delta)]\, n\, \Delta e^{\mathscr{B}(T, 0)} + (A^j_0(0) + O(\Delta))\, m\nu\}\; e^{\mathscr{A}(0, T)} \tag{A2.8}$$

for $j = 0, 1, 2$ *and*

$$D_\mu \hat{a}(\mu, \zeta, \nu)(0) = \nu^{-1}\{[A_0(T) - B_0(T) + O(e^{-\delta T}) + O(\Delta)]\, n\, \Delta e^{\mathscr{B}(T, 0)} + O(\nu)\}\; e^{\mathscr{A}(0, T)}, \tag{A2.9}$$

$$D_\nu \hat{a}(\mu, \xi, \nu)(0) = O(e^{-\delta T}), \tag{A2.10}$$

with T *given by* (A2.7) *and* $\delta > 0$.

Note that the assumption on T to be sufficiently large gives a lower bound on $\mu - \zeta$ for any fixed $\nu \leqslant \nu_0$.

Proof. Following [9] we construct the solution $(\hat{a}, \hat{b})$ of the boundary value problem (A2.4)–(A2.6) as a perturbation of the solution

$$a(t) = 0, \qquad b(t) = q(t) \tag{A2.11}$$

of (A2.4), (A2.5). To this end we consider the linearization of (A2.4), (A2.5) along its solution (A2.11), i.e.,

$$\frac{d}{dt}(\delta a) = A_0(t)\, \delta a \tag{A2.12}$$

$$\frac{d}{dt}(\delta b) = D_a B(0, q(t))\, q(t)\, \delta a + \tilde{B}(t)\, \delta b. \tag{A2.13}$$

Denote

$$\Phi(t, \tau) = \begin{pmatrix} \Phi_{aa}(t, \tau) & \Phi_{ab}(t, \tau) \\ \Phi_{ba}(t, \tau) & \Phi_{bb}(t, \tau) \end{pmatrix}$$

the transition matrix of (A2.12), (A2.13). We have

$$\Phi_{aa}(t,\tau)=e^{\mathscr{A}(t,\tau)}, \qquad \Phi_{ab}(t,\tau)=0, \tag{A2.14}$$

$$\begin{aligned}\Phi_{ba}(t,\tau)&=\int_\tau^t e^{\tilde{\mathscr{B}}(t,s)}\,D_aB(0,q(s))\,q(s)\,e^{\mathscr{A}(s,\tau)}ds,\\ \Phi_{bb}(t,\tau)&=e^{\tilde{\mathscr{B}}(t,\tau)}.\end{aligned} \tag{A2.15}$$

By (A2.2) and (A2.3), we have

$$|B_0(t)-\tilde{B}(t)|<\Delta e^{-\beta t}, \tag{A2.16}$$

hence

$$\begin{aligned}|\tilde{\mathscr{B}}(t,\tau)-\mathscr{B}(t,\tau)|&\leqslant\frac{\Delta}{\beta}\,|e^{-\beta\tau}-e^{-\beta t}|, &&\text{for}\quad t\geqslant\tau\geqslant 0,\\ |\mathscr{B}(t,\tau)-\tilde{\mathscr{B}}(t,\tau)|&\leqslant\Delta|t-\tau|, &&\text{for}\quad t,\tau\geqslant 0,\end{aligned}$$

and

$$|\Phi_{ba}(t,\tau)|=O(e^{\mathscr{B}(t,0)+\eta\,|t-\tau|})\int_\tau^t e^{\mathscr{A}(s,\tau)}\,ds. \tag{A2.17}$$

Denote

$$P_1(t)=\Phi(t,0)\,P_a\Phi(0,t), \qquad P_2(t)=\Phi(t,0)\,P_b\Phi(0,t),$$
$$\Phi_j(t,\tau)=\Phi(t,\tau)\,P_j(\tau)=P_j(t)\,\Phi(t,\tau), \qquad \text{for}\quad j=1,2,$$

where $P_a(a,b)=(a,0)$, $P_b(a,b)=(0,b)$.

A straightforward computation using (A2.17) yields

$$\begin{aligned}\Phi_1(t,\tau)&=\begin{pmatrix}\Phi_{aa}(t,\tau) & 0\\ \Phi_{ba}(t,0)\,\Phi_{aa}(0,\tau) & 0\end{pmatrix}\\ &=e^{\mathscr{A}(t,\tau)}[1+O(\Delta e^{\mathscr{B}(t,0)+\eta t})]\,P_a,\end{aligned} \tag{A2.18}$$

$$\begin{aligned}\Phi_2(t,\tau)&=\begin{pmatrix}0 & 0\\ \Phi_{bb}(t,0)\,\Phi_{ba}(0,\tau) & \Phi_{bb}(t,\tau)\end{pmatrix}\\ &=O(\Delta e^{\mathscr{B}(t,0)+\eta t})\,P_a+e^{\tilde{\mathscr{B}}(t,\tau)}P_b,\end{aligned} \tag{A2.19}$$

and, because of (A2.16),

$$|\Phi_1(t,\tau)|=O(e^{\mathscr{A}(t,\tau)}), \qquad \text{for}\quad 0\leqslant t\leqslant\tau, \tag{A2.20}$$

$$|\Phi_2(t,\tau)|=O(e^{\mathscr{B}(t,\tau)+\eta(t-\tau)}), \qquad \text{for}\quad 0\leqslant\tau\leqslant t \tag{A2.21}$$

(cf. also [9, Lemma 2]), where η can be made arbitrarily small by the choice of $\varDelta$.

Also, we have

$$P_1(t) = \begin{pmatrix} 1 & 0 \\ \varPhi_{ba}(t,0)\,\varPhi_{aa}(0,t) & 0 \end{pmatrix} = P_a + \begin{pmatrix} 0 & 0 \\ O(\varDelta e^{\mathscr{B}(t,0)+\eta t}) & 0 \end{pmatrix}, \tag{A2.22}$$

$$P_2(t) = \begin{pmatrix} 0 & 0 \\ \varPhi_{bb}(t,0)\,\varPhi_{ba}(0,t) & 1 \end{pmatrix} = P_b + \begin{pmatrix} 0 & 0 \\ O(\varDelta e^{\mathscr{B}(t,0)+\eta t}) & 0 \end{pmatrix}, \tag{A2.23}$$

with an arbitrary small $\eta > 0$.

Fix $\eta > 0$ and denote U, V the space $C([0,T],\mathbb{R})$ endowed by the norm

$$\|u\|_U = \sup_{0 \leqslant t \leqslant T} e^{-\mathscr{A}(t,T)+\eta(t-T)}\,|u(t)|,$$

$$\|v\|_V = \sup_{0 \leqslant t \leqslant T} e^{-\mathscr{B}(t,0)-2\eta t}\,|v(t)|,$$

respectively, and

$$W = U \times V.$$

For $Y \in C([0,T],\mathbb{R}^2)$, denote by $|Y|$ its supremum norm and

$$\|Y\| = \|P_1(\cdot)\,Y(\cdot)\|_U + \|P_2(\cdot)\,Y(\cdot)\|_V.$$

Note that $|Y| \leqslant \|Y\|$.

We have the following

A2.2. Lemma. *The norms $\|Y\|$ and $\|P_a Y\|_U + \|P_b Y\|_V$ are equivalent in W. More precisely, for $Y = (u, v) \in C([0,T],\mathbb{R}^2)$ we have*

$$(1+\varDelta)^{-1}\,\|u\|_U \leqslant \|P_1 Y\|_U \leqslant (1+\varDelta)\,\|u\|_U, \tag{A2.24}$$

$$\|P_2 Y\|_V \leqslant \|v\|_V + O(|u|), \tag{A2.25}$$

and

$$\|v\|_V \leqslant \|P_2 Y\|_V + O(|P_1 Y|). \tag{A2.26}$$

Proof. The equivalence of the norms follows immediately from (A2.24)–(A2.26). The estimate (A2.24) follows immediately from (A2.22) and so does (A2.25) from (A2.23). To obtain (A2.26) we note that from (A2.23) it follows

$$|v(t)| \leqslant |P_2(t)\,Y(t)| + O(\varDelta e^{\mathscr{B}(t,0)}\,P_1 Y(t)),$$

which means

$$\|v(\,.\,)\|_V \leqslant \|P_2 Y\|_V + O(|P_1 Y|) = \|P_2 Y\| + (\|Y\|_U). \quad \blacksquare$$

To continue the proof of Proposition A2.1 we introduce the time-dependent transformation

$$a = u, \qquad b = q(t) + v.$$

Then, a, b is a solution of (A2.4), (A2.5) if and only if u, v satisfy the system of equations

$$\dot{u} = A_0(t)\, u + g_u(t, u, v), \tag{A2.27}$$

$$\dot{v} = D_a B(0, q(t), \zeta + vt)\, q(t)\, u + \tilde{B}(t)\, v + g_v(t, u, v), \tag{A2.28}$$

where

$$g_u(t, u, v) = A(u, q(t) + v)\, u - A_0(t)\, u,$$

$$g_v(t, u, v) = B(u, q(t) + v)(q(t) + v) - B(0, q(t))\, q(t)$$
$$- D_a B(0, q(t))\, q(t)\, u - \tilde{B}(t)\, v.$$

Denote $Y = (u, v)$, $g = (g_u, g_v)$. From the definition of g it follows immediately

$$|D_\zeta^j g_u(t, Y)| = O(|Y|\ |u|), \tag{A2.29}$$

$$|D_\zeta^j g_v(t, Y)| = O(|Y|\ |v| + |q(t)|\ |Y|^2), \tag{A2.43}$$

hence

$$\|D_\zeta^j g_u(\cdot, Y(\cdot))\|_U = O(|Y|\ \|u\|_U), \tag{A2.30}$$

$$\|D_\zeta^j g_v(\cdot, Y(\cdot))\|_V = |Y|\ O(\|v\|_V + \|q\|_V\, |Y|) \tag{A2.45}$$

for $j = 0, 1, 2$. Further, we have

$$|D_Y g(t, Y)| = O(|Y|), \tag{A2.31}$$

$$|D_v g_u(t, Y)| = O(|u|), \tag{A2.32}$$

$$|D_u g_v(t, Y)| = O(|q(t)|\ |u| + |v|). \tag{A2.33}$$

The functions $\hat{a}(t)$, $\hat{b}(t)$ solve the boundary value problem (A2.4)–(A2.6) if and only if $\hat{Y}(t) = (\hat{u}(t), \hat{v}(t))$ solves (A2.27)–(A2.28) as well as

$$v(0) = 0, \tag{A2.34}$$

$$u(T) = mv + n(q(T) + v(T)) + \Omega(q(T) + v(T)). \tag{A2.35}$$

We obtain the solution of (A2.27), (A2.28), (A2.34), (A2.35) as the fixed point of the C^2-operator $\mathscr{T}: W \to W$ defined by

$$\begin{aligned}\mathscr{T}(Y) = &\int_0^t \Phi_2(t, s)\, g(s, Y(s))\, ds + \int_T^t \Phi_1(t, s)\, g(s, Y(s))\, ds \\ &+ \Phi_1(t, T)[[mv + n(q(T) + v(T)) + \Omega(q(T) + v(T))]\, P_a \\ &+ (q(T) + v(T))\, P_b].\end{aligned}$$

It can be readily checked that $\hat{Y}$ is a solution of the BVP (A2.27), (A2.28), (A2.34), (A2.35) if and only if it is a fixed point of $\mathscr{T}$.

By [9], the Nemytski operator $G: W \to W$ defined by $G(Y)(t) = g(t, Y)$ is a C^2 function of ζ, μ. Rescaling time as in Proposition 4 of [9] one verifies that G is a C^2 function of T as well.

We prove that for Δ, $v \leqslant v_0$ sufficiently small and $T \geqslant T_0$ sufficiently large there exists a $\rho > 0$ such that $\mathscr{T}$ is a contraction in $\|Y\| \leqslant \rho$ uniform in ζ, μ and T. Then, by the uniform contraction principle, $\mathscr{T}$ has a unique fixed point $\hat{Y} = (\hat{u}, \hat{v})$ in $\|\hat{Y}\| \leqslant \rho$ which is a C^2 function of ζ, μ and T.

To obtain the necessary estimates we first turn the estimates (A2.29), (A2.30) and (A2.31)–(A2.33) of g into those of its P_1- and P_2-projections. We have

$$|D_\zeta^j P_1 g(t, Y)| = |1 + O(\Delta e^{\mathscr{B}(t,0) + \eta t})|\, |D_\zeta^j g(t, Y)| = O(|Y| \cdot |u|), \tag{A2.36}$$

$$\begin{aligned}|D_\zeta^j P_2 g(t, Y)| &= O(\Delta e^{\mathscr{B}(t,0) + \eta t})|\, D_\zeta^j g_u(t, Y)| + |D_\zeta^j g_v(t, Y)| \\ &= O(\Delta e^{\mathscr{B}(t,0) + \eta t})\, |Y|\, |u| + |Y|\, |v| + |q(t)|\, |Y|^2 \\ &= O(\Delta e^{\mathscr{B}(t,0) + \eta t})\, |Y|^2 + |Y|\, |v|,\end{aligned} \tag{A2.37}$$

hence, for $Y = (u, v) \in W$,

$$\|D_\zeta^j P_1 g(t, Y)\|_U = O(|Y|\, \|u\|_U), \tag{A2.38}$$

$$\|D_\zeta^j P_2 g(t, Y)\|_V = O(|Y|^2 + |Y|\, \|v\|_V) \tag{A2.39}$$

for $j = 0, 1, 2$, as well as

$$|P_1 D_v g(t, Y)| = O(D_v g_u(t, Y)) = O(|u|), \tag{A2.40}$$

$$\begin{aligned}|P_2 D_u g(t, Y)| &= O(\Delta e^{\mathscr{B}(t,0) + \eta t})|\, D_u g_u(t, Y)| + |D_u g_v(t, Y)| \\ &= O(\Delta e^{\mathscr{B}(t,0) + \eta t})\, |Y| + O(\Delta e^{\mathscr{B}(t,0) + \eta t})\, |u| + |v| \\ &= O(\Delta e^{\mathscr{B}(t,0) + \eta t})\, |Y| + |v|.\end{aligned} \tag{A2.41}$$

We have

$$\|\mathscr{T}(Y)\| \leqslant I_1 + I_2,$$

where

$$I_1 = \sup_{0 \leqslant t \leqslant T} e^{-\mathscr{A}(t, T) + \eta(t-T)} \Big| \Phi_1(t, T)[[mv + n(q(T) + v(T)) + \Omega(q(T) + v(T))] P_a + (q(T) + v(T)) P_b] + \int_T^t \Phi_1(t, s)\, g(s, Y(s))\, ds \Big|,$$

$$I_2 = \sup_{0 \leqslant t \leqslant T} e^{-\mathscr{B}(t, 0) - 2\eta t} \left| \int_0^t \Phi_2(t, s)\, g(s, Y(s))\, ds \right|. \tag{A2.42}$$

Because of (A2.18) and (A2.36) we have

$$\begin{aligned} I_1 &\leqslant \sup_{0 \leqslant t \leqslant T} e^{-\mathscr{A}(t, T) + \eta(t-T)} O\Big(|\Phi_1(t, T)|\, |mv + n(q(T) + v(T)) \\ &\quad + \Omega(q(T) + v(T))| + \int_{\mathscr{T}}^t |\Phi_1(t, s)|\, |g_u(s, Y(s))|\, ds \Big) \\ &= O(v) + e^{\mathscr{B}(T, 0) + \eta T} O(\|q\|_V + \|v\|_V) \\ &\quad + \sup_{0 \leqslant t \leqslant T} \int_{\mathscr{T}}^t e^{\eta(t-s)} O(e^{-\mathscr{A}(s, T) - \eta(T-s)})(|u(s)|\, |Y(s)|)\, ds \\ &= O(v) + e^{\mathscr{B}(T, 0) + \eta T} O(\|q\|_V + \|v\|_V) + O(|Y|\, \|u\|_U). \end{aligned} \tag{A2.43}$$

Further, because of (A2.18) and (A2.37) we have

$$\begin{aligned} I_2 &\leqslant \sup_{0 \leqslant t \leqslant T} e^{\mathscr{B}(t, 0) - 2\eta t} \int_0^t |\Phi_2(t, s)|\, |P_2(s)\, g(s)\, Y(s)|\, ds \\ &= \sup_{v \leqslant t \leqslant T} e^{-\mathscr{B}(t, 0) - 2\eta t} \int_0^t O((e^{\mathscr{B}(t, s) + \eta(t-s)}) \\ &\quad \times (\Delta e^{\mathscr{B}(s, 0) + \eta s}\, |y^2(s)| + |Y(s)|\, |v(s)|))\, ds \\ &= O(|Y|^2 + |Y|\, \|v\|_V) = |Y|\, O(\|Y\|). \end{aligned} \tag{A2.44}$$

It is obvious from (A2.43), (A2.44) that, if Δ, v are chosen sufficiently small and T_0 is chosen sufficiently large, then a sufficiently small ball $\|u\|_U + \|v\|_V \leqslant \rho$ is mapped into itself by $\mathscr{T}$ for $v \leqslant v_0$ and $T \geqslant T_0$.

Further, for $Y_1=(u_1, v_1)\in W$, $Y_2=(u_2, v_2)\in W$ we have

$$\|\mathscr{T}(Y_1)-\mathscr{T}(Y_2)\|\leqslant J_1+J_2,$$

where

$$J_1\leqslant \sup_{0\leqslant t\leqslant T} e^{-\mathscr{A}(t,T)+\eta(t-T)}\Big\{|\Phi_1(t,T)|\,[n\,|v_1(T)-v_2(T)|$$
$$+|\Omega(q(T)+v_1(T))-\Omega(q(T)+v_2(T))|]$$
$$+\int_T^t |\Phi_1(t,s)|\,|g(s, Y_1(s))-g(s, Y_2(s))|\,ds\Big\},$$

$$J_2\leqslant \sup_{0\leqslant t\leqslant T} e^{-\mathscr{B}(t,0)-2\eta t}\int_0^t |\Phi_2(t,s)|\,|g(s, Y_1(s))-g(s, Y_2(s))|\,ds.$$

Because of (A2.18) and (A2.40), (A2.41) we have

$$J_1=\sup_{0\leqslant t\leqslant T} e^{\mathscr{A}(t,T)+\eta(t-T)}\Big\{O((\Phi_1(t,T))\,|v_1(T)-v_2(T)|)$$
$$+\int_T^t |\Phi_1(t,s)|\,|P_1(g(s, Y_1(s))-g(s, Y_2(s))|\,ds\Big\}$$
$$=O(e^{\mathscr{B}(T,0)+\eta T})(\|v_1-v_2\|_V)$$
$$+\sup_{0\leqslant t\leqslant T}\int_T^t e^{\eta(t-s)}[e^{-\mathscr{A}(s,T)-\eta(T-s)}O(|u_1(s)|+|u_2(s)|)\,|v_1(s)-v_2(s)|$$
$$+O(|Y_1(s)|+|Y_2(s)|)\,|u_1(s)-u_2(s)|]$$
$$=O(e^{\mathscr{B}(T,0)+\eta T}+\|u_1\|_U+\|u_2\|_U)\,\|v_1-v_2\|_V$$
$$+O(|Y_1|+|Y_2|)\,\|u_1-u_2\|_U, \tag{A2.45}$$

and, because of (A2.19), (A2.40)–(A2.42),

$$J_2\leqslant \sup_{0\leqslant t\leqslant T}\int_0^t e^{-2\eta(t-s)}e^{-\mathscr{B}(s,0)-2\eta s}\,|P_2(g(s, Y_2(s))-g(s, Y_2(s)))|\,ds$$
$$=|Y_1+Y_2|\,O(\|v_1-v_2\|_V)+(|Y_1|+|Y_2|+\|v_1+v_2\|_V)\,\|u_1-u_2\|_U. \tag{A2.46}$$

From (A2.45) and (A2.46) it follows that if ρ is chosen small enough then $\mathscr{T}$ is a contraction in $\|u\|_U+\|v\|_V\leqslant\rho$ which is uniform in $\mu\in[\gamma_0-\kappa, \gamma_0+\kappa]$, $\zeta\in[\gamma_0-2\kappa, \mu)$, $v\leqslant v_0$ and $T\geqslant T_0$. By the uniform contraction principle [4] there is a unique fixed point $\hat{Y}(\mu, \zeta, T)=(\hat{u}(\mu, \zeta, T),$

$\hat{v}(\mu, \zeta, T))$ in $\|Y\| \leqslant \rho$ which is a C^2 function of μ, ζ, T, v for $v \leqslant v_0$ sufficiently small, $\mu \in [\gamma_0 - \kappa, \gamma_0 + \kappa]$, $\zeta \in [\gamma_0 - 2\kappa, \mu)$, $v \leqslant v_0$ and $T \geqslant T_0$.

Moreover, we have

$$\|\hat{Y}\| = O(\|\mathcal{T}(0)\|),$$

i.e.,

$$\|\hat{Y}\| = \|\Phi_1(0, \cdot)\|_U [|m| v + 2\bar{n} |q(T)|] \Omega = O(v + \Delta e^{\mathscr{B}(T, 0)}) \tag{A2.47}$$

and, consequently,

$$|\hat{v}(T)| \leqslant e^{\mathscr{B}(T, 0) + \eta T} \|\hat{v}\| = O(e^{\mathscr{B}(T, 0) + \eta T} v + \Delta e^{2\mathscr{B}(T, 0) + 2\eta T}). \tag{A2.48}$$

To obtain the estimates (A2.8)–(A2.9) we first note that

$$\begin{aligned} &\hat{u}(\mu, \zeta, T)(0) \\ &\quad = \Phi_1(0, T)[[mv + n(q(T) + \hat{v}(T)) + \Omega(q(T) + \hat{v}(T))] P_a \\ &\qquad + (q(T) + \hat{v}(T)) P_b] + \int_T^0 \Phi_1(0, s)\, g(s, \hat{Y}(s))\, ds. \end{aligned} \tag{A2.49}$$

By (4.8), (A2.24), (A2.36), (A2.47), (A2.48) we have for $v \leqslant v_0$, $\mu - \zeta \geqslant T_0$

$$\begin{aligned} &|\hat{u}(0) - (n\, \Delta e^{\mathscr{B}(T, 0)} - mv)\, e^{\mathscr{A}(0, T)}| \\ &\quad = e^{\mathscr{A}(0, T)}[o(v) + O(\Delta e^{2\mathscr{B}(T, 0) + 2\eta T})] + \int_T^0 e^{\mathscr{A}(0, s)} O(|Y(s)|\, |u(s)|)\, ds \\ &\quad = e^{\mathscr{A}(0, T)}[o(v) + O(\Delta e^{2\mathscr{B}(T, 0) + 2\eta T})] + T(O(v) + O(\Delta e^{\mathscr{B}(T, 0)})^2 \\ &\quad = e^{\mathscr{A}(0, T)}[o(v) + O(\Delta e^{2B(T, 0) + 2\eta T})] \end{aligned}$$

which gives (A2.8) for $j = 0$ with $\delta = 2(\beta - \eta) > 0$.

To obtain (A2.8) for $j = 1$ we differentiate (A2.49) with respect to ζ:

$$\begin{aligned} D_\zeta \hat{u}(0) = {} & D_\zeta \Phi_1(0, T)[(mv + n(q(T) + \hat{v}(T))) + \Omega(q(T) \\ & + \hat{v}(T)) P_a + (q(T) + \hat{v}(T)) P_b] + \Phi_1(0, T) \\ & \times [n(D_\zeta q(T) + D_\zeta \hat{v}(T)) + D_b \Omega(q(T) + \hat{v}(T)) D_\zeta (q(T) \\ & + \hat{v}(T)) P_a + D_\zeta (q(T) + \hat{v}(T)) P_b] + \int_T^0 D_\zeta \Phi_1(0, s)\, g(s, \hat{Y}(s))\, ds \\ & + \int_T^0 \Phi_1(0, s)[D_\zeta g(s, \hat{Y}(s)) + D_Y g(s, \hat{Y}(s))\, D_\zeta \hat{Y}(s)]\, ds. \end{aligned} \tag{A2.50}$$

To determine the leading terms of $D_\zeta \hat{u}(0)$ we need estimates for the factors of the terms of (A2.50). We have

$$\frac{d}{dt}(D_\zeta q) = \tilde{B}(t)\, D_\zeta q + D_\xi B(0, q(t), \zeta + vt)\, q(t),$$

$$= \tilde{B}(t)\, D_\zeta q + O(\Delta e^{\mathscr{B}(t,0)})$$

and

$$D_\zeta q(0) = 0,$$

hence

$$|D_\zeta q(t)| = O\left(\Delta \int_0^t e^{\mathscr{B}(t,s)} e^{\mathscr{B}(s,0)}\, ds\right) = O(\Delta e^{\mathscr{B}(t,0)+\eta t}). \tag{A2.51}$$

To estimate $D_\zeta \Phi_1$, $D_\zeta \Phi_2$ we first estimate $D_\zeta \Phi_{aa}$, $D_\zeta \Phi_{bb}$. By definition (A2.14) of Φ_{aa} we have

$$D_\zeta \Phi_{aa}(t, \tau) = D_\zeta e^{\mathscr{A}(t,\tau)} = e^{\mathscr{A}(t,\tau)}\, D_\zeta \int_\tau^t A_0(s, \zeta)\, ds$$

$$= \frac{1}{v}\,[A_0(t) - A_0(\tau) + O(\Delta)]\, e^{\mathscr{A}(t,\tau)} \tag{A2.52}$$

for $0 \leqslant t \leqslant \tau$, since

$$D_\zeta \int_\tau^t A(0, q(s, \zeta), \zeta + vs)\, ds$$

$$= \frac{1}{v} D_\zeta \int_{\zeta + v\tau}^{\zeta + vt} A\left(0, q\left(\frac{\xi - \zeta}{v}, \zeta\right), \xi\right) d\xi$$

$$= \frac{1}{v}\left[A_0(t) - A_0(\tau) + \int_{\zeta + v\tau}^{\zeta + vt} \left|D_b A\left(0, q\left(\frac{\xi - \zeta}{v}, \zeta\right), \xi\right)\right| \left|D_\zeta q\left(\frac{\xi - \zeta}{v}\right)\right|\right.$$

$$\left. + \frac{1}{v}\left|B_0\left(\frac{\xi - \zeta}{v}\right) q\left(\frac{\xi - \zeta}{v}, \zeta\right)\right| d\xi\right]$$

$$= \frac{1}{v}\,[A_0(t) - A_0(\tau) + O(\Delta)] \tag{A2.53}$$

by (A2.51).

Similarly we obtain

$$D_\zeta \Phi_{bb}(t, \tau) = D_\zeta e^{\tilde{\mathscr{B}}(t, \tau)} = e^{\mathscr{B}(t, 0) + \eta t}[B_0(t) - B_0(\tau) + O(\Delta)] \quad \text{(A2.54)}$$

for $0 \leqslant \tau \leqslant t$.

Further, by (A2.15), (A2.51), (A2.8) for $j = 0$ and (A2.54) we have

$$\begin{aligned}
|D_\zeta(\Phi_{ba}(t, 0)\, \Phi_{aa}(0, \tau)| &= O\left(\Delta \int_0^t D_\zeta(e^{\tilde{\mathscr{B}}(t, s)})\, e^{\mathscr{B}(s, 0)} e^{\mathscr{A}(s, \tau)}\, ds\right) \\
&\quad + O\left(\Delta \int_0^t e^{\tilde{\mathscr{B}}(t, s)}\, D_\zeta(e^{\tilde{\mathscr{B}}(s, 0)})\, e^{\mathscr{A}(s, \tau)}\, ds\right) \\
&\quad + O\left(\Delta \int_0^t e^{\tilde{\mathscr{B}}(t, s)} e^{\tilde{\mathscr{B}}(s, 0)} D_\zeta e^{\mathscr{A}(s, \tau)}\, ds\right) \\
&= O\left(\frac{1}{\nu} \Delta t e^{\tilde{\mathscr{B}}(t, 0)} e^{\mathscr{A}(t, \tau)}\right) \\
&= O\left(\frac{1}{\nu} \Delta e^{\mathscr{B}(t, 0) + \eta t} e^{\mathscr{A}(t, \tau)}\right) \qquad \text{(A2.55)}
\end{aligned}$$

for $0 \leqslant t \leqslant \tau$, and, similarly, by (A2.19), (A2.11) for $j = 0$ and (A2.51),

$$|D_\zeta(\Phi_{bb}(t, 0)\, \Phi_{ba}(0, \tau))| = O\left(\frac{\Delta}{\nu}\, e^{\tilde{\mathscr{B}}(t, 0) + \eta t}\right) \qquad \text{(A2.56)}$$

for $0 \leqslant \tau \leqslant t$.

From (A2.53)–(A2.56) and (A2.18), (A2.19) we obtain

$$D_\zeta \Phi_1(t, \tau) = \frac{1}{\nu}\, [A_0(t) - A_0(\tau) + O(\Delta)]\, e^{\mathscr{A}(t, \tau)} \qquad \text{(A2.57)}$$

for $t \geqslant \tau \geqslant 0$ and

$$D_\zeta \Phi_2(t, \tau) = \frac{1}{\nu}\, [B_0(t) - B_0(\tau) + O(\Delta)]\, e^{\mathscr{B}(t, \tau) + \eta(t - \tau)} \qquad \text{(A2.58)}$$

for $0 \leqslant t \leqslant \tau$.

To obtain the leading terms of $D_\zeta \hat{u}(0)$, it remains to estimate $\|D_\zeta \hat{Y}\|$. By the uniform contraction principle we have

$$\|D_\zeta \hat{Y}\| = O(\|D_\zeta \mathscr{T}(\hat{Y})\|). \qquad \text{(A2.59)}$$

Using the estimates (A2.17), (A2.5), (A2.6), (A2.38), (A2.39) and (A2.57), (A2.58) we obtain similarly as in (A2.43), (A2.44)

$$
\begin{aligned}
&\|D_\zeta \mathscr{T}(\hat{Y})\| \\
&\quad = \left\| \int_T^{\cdot} [D_\zeta \Phi_2(\cdot, s)\, g(s, \hat{Y}(s)) + \Phi_2(\cdot, s)(D_\zeta g(s, \hat{Y}(s))]\, ds \right\|_V \\
&\qquad + \left\| \int_T^{\cdot} [D_\zeta \Phi_1(\cdot, s)\, g(s, \hat{Y}(s)) + \Phi_2(\cdot, s)\, D_\zeta g(s, \hat{Y}(s))]\, ds \right\|_U \\
&\qquad + \|D_\zeta \Phi(\cdot, T)[mv + n(q(T) + \hat{v}(T)) + \Omega(q(T) + \hat{v}(T))]\|_U \\
&\qquad + \|\Phi(\cdot, T)[D_\zeta q(T) + D_b \Omega(q(T) + \hat{v}(T))\, D_\zeta q(T)]\|_U \\
&\quad = \frac{1}{v} O[v + e^{\mathscr{B}(T,0) + \eta T}(\|q\|_V + \|\hat{v}\|_V) + |Y|\, (\|\hat{u}\|_U + \|\hat{v}\|_V + \|q\|_V\, |\hat{Y}|)] \\
&\quad = \frac{1}{v} O(e^{\mathscr{B}(T,0) + \eta T} + v),
\end{aligned}
$$

hence

$$
\|D_\zeta \hat{Y}\| = \frac{1}{v} O(e^{\mathscr{B}(T,0) + \eta T} + v).
$$

Now, from (A2.50), (A2.3), (A2.19), (A2.21), (A2.38), (A2.39), (4.8), (A2.51), (A2.37), (A2.58), (A2.59) we obtain

$$
\begin{aligned}
D_\zeta \hat{u}(0) = \frac{1}{v} \{ [A_0(0) - A_0(T) + O(\Delta)][mv + n\Delta e^{\mathscr{B}(T,0)} + o(v)] \\
+ O(e^{2B(T,0) + 2\eta T}))\}\, e^{\mathscr{A}(0,T)}.
\end{aligned}
\tag{A2.60}
$$

We have

$$
\hat{a}(\mu, \zeta)(0) = \hat{u}\left(\mu, \zeta, \frac{\mu - \zeta}{v}\right)(0),
$$

hence, by (A2.60), (A2.47),

$$
D_\zeta \hat{a}(\mu, \zeta)(0) = D_\zeta \hat{u}\left(\mu, \zeta, \mu - \zeta v\right)(0) - \frac{1}{v} D_T \hat{u}\left(\mu, \zeta, \frac{\mu - \zeta}{v}\right)(0), \tag{A2.61}
$$

provided $v \leqslant v_0$ and $\mu - \zeta \geqslant v T_0$.

By (A2.18) we have $\Phi_1(0, T) = e^{\mathscr{A}(0, T)}$. Hence, by (A2.47), (A2.48),

$$
\begin{aligned}
D_T\hat{u}\left(\mu, \zeta, \frac{\mu-\zeta}{v}\right)(0)
&= \{-A_0(T)(mv + n\,\Delta e^{\mathscr{B}(T,0)}) + n\,\Delta B_0(T)\, e^{\mathscr{B}(T,0)} \\
&\quad + O(\Delta e^{2\mathscr{B}(T,0)+2\eta T}) + o(v)\}\, e^{\mathscr{A}(0,T)}. \qquad \text{(A2.62)}
\end{aligned}
$$

By (A2.60)–(A2.62) we obtain

$$
\begin{aligned}
&D_\zeta \hat{a}(\mu, \zeta)(0) \\
&\quad = \frac{1}{v}\{[A_0(0) - B_0(T) + O(\Delta)][n\Delta + O(e^{\mathscr{B}(T,0)+2\eta T})]\, e^{\mathscr{A}(0,T)+\mathscr{B}(T,0)} \\
&\qquad + A_0(0)(mv + o(v))\, e^{\mathscr{A}(0,T)}\} \qquad \text{(A2.63)}
\end{aligned}
$$

with $vT = \mu - \zeta$ which gives (A2.8) for $j = 1$, provided $v \leqslant v_0$ is sufficiently small.

Because $\Phi_1(t, \tau)$ does not depend on μ, computations similar to those leading to (A2.60) give

$$
|D_\mu \hat{u}(0)| = \frac{1}{v}\, O(e^{\mathscr{B}(T,0)+\eta T} + \Delta + v)\, e^{\mathscr{B}(T,0)+\mathscr{A}(0,T)}.
$$

Hence, $D_\mu \hat{u}(0)$ does not contribute to the leading terms of $D_\mu \hat{a}(\mu, \zeta)(0)$. Consequently, we have

$$
\begin{aligned}
D_\mu \hat{a}(\mu, \zeta)(0) &= \frac{1}{v}\{[A_0(T) - B_0(T) + O(\Delta)][n\Delta + O(e^{\mathscr{B}(T,0)+2\eta T}] \\
&\qquad \times e^{\mathscr{B}(T,0)+\mathscr{A}(0,T)} + O(v)\} e^{\mathscr{A}(0,T)} \qquad \text{(A2.64)}
\end{aligned}
$$

with $vT = \mu - \zeta$ which gives (A2.9).

Having chosen $\Delta, \eta, v \leqslant v_0$ small enough and $T \geqslant v(\mu - \zeta)$ large enough we obtain (A2.8), (A2.9) for $j = 1$ from (A2.63), (A2.64), respectively.

It should be obvious from the computations leading to the estimate (A2.64) that up to terms of order

$$
O(v)\, e^{\mathscr{A}(0,T)}\, \Delta + O(\Delta + e^{\mathscr{B}(T,0)+2\eta T}) e^{\mathscr{B}(T,0)+\mathscr{A}(0,T)}
$$

we have

$$
D_\zeta^2 \hat{a}(\mu, \zeta)(0) = n\,\Delta D_\zeta^2 e^{\mathscr{A}(0,(\mu-\zeta)/v)+\mathscr{B}((\mu-\zeta)/v, 0)}
$$

and, therefore,

$$D_\zeta^2 \hat{a}(\mu, \zeta)(0) = \frac{n\Delta}{\nu^2} \{[(A_0(0) - B_0(T))^2 + O(\Delta) + e^{\mathscr{B}(T,0)+2\eta T}] \times e^{\mathscr{B}(T,0)} + O(\nu)\} \, e^{\mathscr{A}(0,T)}.$$

To prove the estimate (A2.10) on $D_\nu \hat{a}$ we proceed similarly as in the case of $D_\zeta \hat{a}$.

First we note that estimates (A2.36)–(A2.37) extend to $D_\nu g$. Then, similarly as for the derivatives with respect to ζ, (A2.51)–(A2.60) we obtain

$$|D_\nu q(t)| = O(\Delta e^{\mathscr{B}(t,0)+\eta t}),$$

$$|D_\nu \Phi_1(t, \tau)| = O\left(\frac{\Delta}{\nu} e^{\mathscr{A}(t,\tau)}\right) \qquad \text{for} \quad 0 \leqslant t \leqslant \tau,$$

$$|D_\nu \Phi_2(t, \tau)| = O\left(\frac{1}{\nu} e^{\mathscr{B}(t,\tau)+\eta(t-\tau)}\right) \qquad \text{for} \quad 0 \leqslant \tau \leqslant t,$$

hence

$$\|D_\nu \hat{Y}\| = \frac{1}{\nu} O(e^{\mathscr{B}(T,0)+\eta T} + \nu),$$

which gives

$$\|D_\nu \hat{u}(0)\| = \nu^{-1} O(e^{-\alpha T} + e^{-(\beta-\eta)T}) = O(e^{-\delta t}),$$

where $\delta < \min\{\alpha, \beta - \eta\} > 0$, provided η has been chosen sufficiently small. Moreover,

$$|D_T \hat{u}(0)| = O(e^{\alpha T}),$$

hence

$$|D_\nu \hat{a}(0)| \leqq |D_\nu \hat{u}(0)| + \frac{\mu - \zeta}{\nu^2} |D_T \hat{u}(0)|$$
$$= O(e^{-\delta T} + \nu^{-2} e^{-\alpha T}) = O(e^{-\delta T}). \quad \blacksquare$$

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